Gov 2001 Section 8, 2025

Delta Method Practice (Hypothesis testing to be uploaded later)

Delta method: Suppose that

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} \mathcal{N}(0, \omega^2)$$

as $n \to \infty$ and g is a continuously differentiable function. Then as $n \to \infty$,

$$\sqrt{n}(g(\hat{\theta}) - g(\theta)) \xrightarrow{d} \mathcal{N}\left(0, \left(\frac{\partial g(\theta)}{\partial \theta}\right)^2 \omega^2\right)$$

Notice that the asymptotic distribution should not contain n. Or we can write as for large n,

$$g(\hat{\theta}) \dot{\sim} \mathcal{N}\left(g(\theta), \frac{(g'(\theta))^2\, \omega^2}{n}\right)$$

Example: If we have $X_{1:n} \overset{i.i.d}{\sim} \mathcal{N}(\mu, \sigma^2)$, and thus $\bar{X}_n \dot{\sim} \mathcal{N}(\mu, \frac{\sigma^2}{n})$, then for large n,

$$g(\bar{X}_n) \dot{\sim} \mathcal{N}\left(g(\mu), (g'(\mu))^2 \frac{\sigma^2}{n}\right)$$

Practice 1: Assume $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} \mathcal{N}(0, v^2)$.

(a) Use the Delta method to find the asymptotic distribution of $\hat{\theta}^4$.

Solution:

Let $g(\hat{\theta}) = \hat{\theta}^4$. Then $g'(\theta) = 4\theta^3$.

By the Delta method:

$$\sqrt{n}(g(\hat{\theta}) - g(\theta)) \xrightarrow{d} \mathcal{N}(0, [g'(\theta)]^2 v^2) = \mathcal{N}(0, 16\theta^6 v^2).$$

So,

$$\hat{\theta}^4 \dot{\sim} \mathcal{N}\left(\theta^4, \frac{16\theta^6 v^2}{n}\right).$$

(b) Use the Delta method to find the asymptotic distribution of

$$\frac{1}{1 + \exp(-\hat{\theta})}.$$

Solution:

Let $g(\hat{\theta}) = \dfrac{1}{1 + \exp(-\hat{\theta})}$. Then this is the logistic function, and its derivative is:

$$g'(\theta) = \frac{e^{-\theta}}{(1 + e^{-\theta})^2} = g(\theta)(1 - g(\theta)).$$

By the Delta method:

$$\sqrt{n}(g(\hat{\theta}) - g(\theta)) \xrightarrow{d} \mathcal{N}(0, [g'(\theta)]^2 v^2).$$

So,

$$\frac{1}{1 + \exp(-\hat{\theta})} \dot{\sim} \mathcal{N}\left(\frac{1}{1 + \exp(\theta)}, \frac{e^{-2\theta} v^2}{n(1 + e^{-\theta})^4}\right).$$

(c) Explain how you can approximately derive the Delta method using the Taylor polynomials. The first-order Taylor approximation of $f(x)$ at value c is $f(x) = f(c) + (x - c)f'(c)$.

Solution:

Delta method uses the first-order Taylor expansion of the function $g(\hat{\theta})$ around θ:

$$g(\hat{\theta}) \approx g(\theta) + g'(\theta)(\hat{\theta} - \theta).$$

Multiply both sides by $\sqrt{n}$:

$$\sqrt{n}(g(\hat{\theta}) - g(\theta)) \approx \sqrt{n}g'(\theta)(\hat{\theta} - \theta).$$

Since $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} \mathcal{N}(0, v^2)$, by continuous mapping theorem:

$$g'(\theta)\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} \mathcal{N}(0, [g'(\theta)]^2 v^2).$$

Therefore,

$$\sqrt{n}(g(\hat{\theta}) - g(\theta)) \xrightarrow{d} \mathcal{N}(0, [g'(\theta)]^2 v^2).$$

Practice 2: Let $Y_1, \ldots, Y_n$ be i.i.d. random variables such that $E(Y_i) = 0$, $\mathrm{Var}(Y_i) = 1$, and the fourth moment $E(Y_i^4)$ exists. Also, define

$$S_n = \frac{1}{n}\sum_{i=1}^{n} Y_i^2, \qquad V_n = \frac{1}{n}\sum_{i=1}^{n}(Y_i^2 - 1)^2.$$

The goal is to identify a constant c such that

$$c \cdot \frac{\sqrt{n}\,(\exp(S_n) - e)}{\sqrt{V_n}} \xrightarrow{d} \mathcal{N}(0, 1).$$

(a) What is the approximated distribution of S_n as $n \to \infty$? Your answer can include $\mathrm{Var}(Y_i^2)$ but should not include $E[Y_i^2]$. (Hint: Use CLT...)

Solution:

Since Y_i are i.i.d. with $E(Y_i) = 0$ and $\mathrm{Var}(Y_i) = 1$, we have $E(Y_i^2) = 1$. By CLT,

$$\sqrt{n}(S_n - E(S_n)) = \sqrt{n}\,(S_n - 1) \xrightarrow{d} \mathcal{N}(0, \mathrm{Var}(Y_i^2)).$$

Therefore, S_n is approximately $\mathcal{N}\left(1, \frac{\mathrm{Var}(Y_i^2)}{n}\right)$ for large n.

(b) Show that $\sqrt{V_n} \xrightarrow{p} \sqrt{\mathrm{Var}(Y_i^2)}$. (Hint: LLN and CMT...)

Solution:

Since V_n is the sample mean of i.i.d. $(Y_i^2 - 1)^2$, By LLN,

$$V_n = \frac{1}{n}\sum_{i=1}^{n}(Y_i^2 - 1)^2 \xrightarrow{p} E[(Y_i^2 - 1)^2] = \mathrm{Var}(Y_i^2).$$

Since the square root function is continuous, by the Continuous Mapping Theorem (CMT),

$$\sqrt{V_n} \xrightarrow{p} \sqrt{\mathrm{Var}(Y_i^2)}.$$

(c) Given parts (a) and (b), prove the following asymptotic result. (Hint: Slutsky...)

$$\frac{\sqrt{n} \cdot (S_n - 1)}{\sqrt{V_n}} \xrightarrow{d} \mathcal{N}(0, 1).$$

Solution:

From part (a), $\sqrt{n}(S_n - 1) \xrightarrow{d} \mathcal{N}(0, \mathrm{Var}(Y_i^2))$. From part (b), $\sqrt{V_n} \xrightarrow{p} \sqrt{\mathrm{Var}(Y_i^2)}$. By Slutsky's theorem and the linear (scale) transformation of normal distribution,

$$\frac{\sqrt{n}(S_n - 1)}{\sqrt{V_n}} \xrightarrow{d} \mathcal{N}(0, 1).$$

(d) Find the constant c such that

$$c \cdot \frac{\sqrt{n}\,(\exp(S_n) - e)}{\sqrt{V_n}} \xrightarrow{d} \mathcal{N}(0, 1).$$

(Hint: Delta method!)

Solution:

Let $g(x) = \exp(x)$. Then $g'(x) = \exp(x)$. Applying the Delta method around $x = 1$,

$$\sqrt{n}\,(g(S_n) - g(1)) = \sqrt{n}\,(\exp(S_n) - e) \xrightarrow{d} \mathcal{N}(0, e^2\,\mathrm{Var}(Y_i^2)).$$

To match the variances, set $c = \frac{1}{e}$. Thus,

$$\frac{1}{e} \cdot \frac{\sqrt{n}\,(\exp(S_n) - e)}{\sqrt{V_n}} \xrightarrow{d} \mathcal{N}(0, 1).$$