Gov 2001 Section 3, Feb. 21, 2025

Poisson Distribution

1. Important calculus results: Taylor expansion of the exponential function
$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad e^x \approx 1 + x \text{ for } |x| \text{ small.}$$
2. PMF and support: $X \sim \text{Pois}(\lambda)$, then
$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad \text{for k} \in \{0, 1, 2, \dots\}$$
3. Interpretation: There are rare events (low probability events) that occur at an average rate of λ occurrences per unit space or time. The number of events that occur in that unit of space or time is $X \sim \text{Pois}(\lambda)$.
4. Poisson approximation: $X \sim \text{Bin}(n, p)$ where $n \to \infty$ and $p \to 0$ (alternatively, given the definition of binomial distribution, $X = X_1 + X_2 + \cdots + X_n$ where $X_1, \dots X_n$ approximately follows i.i.d. Bern(p) for $n \to \infty$ and $p \to 0$), then X approximately follows Pois(np).

Continuous Random Variable

1. Definition of CDF is the same as the discrete case: $F(x) = P(X \leq x)$ for a continuous r.v. X, F is an increasing function with $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.
2. The PDF of X is just $f(x) = F'(x)$, and $F(x) = \int_{-\infty}^{x} f(t)\, dt$.
3. Properties of a PDF: it must be nonnegative and integrate to 1: $\int_{-\infty}^{\infty} f(t)\, dt = P(x \in (-\infty, \infty)) = 1$ (axiom of probability).
4. Find the probability that a continuous r.v. takes on a value within interval $[a, b]$:
$$P(a \leq X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a) = \int_a^b f(x)\, dx$$
5. LOTUS for continuous r.v. X with pdf $f(x)$:
$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

Uniform Distribution

PDF of $X \sim \text{Unif}(a, b)$:
$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

CDF of $X \sim \text{Unif}(a, b)$:
$$F(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leq x \leq b, \\ 1, & x > b. \end{cases}$$

$$\mathbb{E}[X] = \frac{a+b}{2}, \quad \text{Var}(X) = \frac{(b-a)^2}{12}$$

Universality of Uniform: for any continuous r.v. X, we have $F(X) \sim \text{Unif}(0, 1)$. For example, for $X \sim \text{Expo}(1)$ with CDF $F(x) = 1 - e^{-x}$, we have $1 - e^{-X} \sim \text{Unif}(0, 1)$. Proof: Let $Y = F(X)$, its CDF is $F(y) = P(Y \leq y) = P(F(X) \leq y) = P(X \leq F^{-1}(y)) = F(F^{-1}(y)) = y$.

Exercise Questions

1. We have $X, Y \overset{\text{i.i.d.}}{\sim} \text{Pois}(\lambda)$, calculate:

   (a) $\mathbb{E}[2^X]$;

   (b) $\mathbb{E}[e^{3Y}]$;

   (c) $P(X > Y)$.

   (a) By LOTUS,

   $$\mathbb{E}[2^X] = \sum_{k=0}^{\infty} \frac{2^k \cdot \lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(2\lambda)^k}{k!} = e^{-\lambda} \cdot e^{2\lambda} = e^{\lambda}$$

   (b) By LOTUS,

   $$\mathbb{E}[e^{3Y}] = \sum_{k=0}^{\infty} \frac{e^{3k} \cdot \lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^3)^k}{k!} = e^{\lambda(e^3-1)}$$

   (c) By symmetry of i.i.d. random variables, we have $P(X < Y) = P(X > Y)$. By axioms of probability, we have $P(X > Y) = \frac{1}{2}[1 - P(X = Y)]$. Now, by LOTP we have

   $$P(X = Y) = \sum_{k=0}^{\infty} P(X = k)P(Y = k) = e^{-2\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(k!)^2}$$

2. Prove the following properties of Poisson distribution:

   (a) $X \sim \text{Pois}(\lambda)$, then $\mathbb{E}[X(X-1)(X-2)\dots(X-r+1)] = \lambda^r$.

   (b) $X \sim \text{Pois}(\lambda)$, then $\text{Var}(X) = \lambda$.

   (c) $X \sim \text{Pois}(\lambda)$, then for any function g, we have $\mathbb{E}[X \cdot g(X)] = \lambda \mathbb{E}[g(X+1)]$.

   (d) If $X \sim \text{Pois}(\lambda_1)$ and $Y \sim \text{Pois}(\lambda_2)$, $X \perp Y$, then $X + Y \sim \text{Pois}(\lambda_1 + \lambda_2)$. Hint: use the Binomial theorem $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$.

   (a) By LOTUS,

   $$\mathbb{E}[X \dots (X-r+1)] = \sum_{k=r}^{\infty} \frac{k!}{(k-r)!} \cdot \frac{\lambda^k e^{-\lambda}}{k!} = \sum_{h=0}^{\infty} \frac{\lambda^{h+r} e^{-\lambda}}{h!} = \lambda^r$$

   Notice that when $k < r$, we have $k \dots (k-r+1) = 0$.

   (b) We can write $X^2 = X(X-1) + X$. Use linearity and the conclusion in (a), we have

   $$\mathbb{E}[X^2] = \mathbb{E}[X(X-1)] + \mathbb{E}[X] = \lambda^2 + \lambda$$

   Therefore, we can calculate the variance:

   $$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

(c) By LOTUS,

$$\mathbb{E}[Xg(X)] = \sum_{k=0}^{\infty} \frac{kg(k)\lambda^k e^{-\lambda}}{k!} = \sum_{k=1}^{\infty} \frac{g(k)\lambda^k e^{-\lambda}}{(k-1)!} = \lambda \sum_{h=0}^{\infty} \frac{g(h+1)\lambda^h e^{-\lambda}}{h!} = \lambda \mathbb{E}[g(X+1)]$$

(d) Let $Z = X + Y$, we have by LOTP,

$$\begin{aligned} P(Z=z) &= \sum_{x=0}^{z} P(X=x)P(Y=z-x) \\ &= \sum_{x=0}^{z} \frac{\lambda_1^x e^{-\lambda_1}}{x!} \cdot \frac{\lambda_2^{z-x} e^{-\lambda_2}}{(z-x)!} \\ &= e^{-(\lambda_1+\lambda_2)} \frac{1}{z!} \sum_{x=0}^{z} \binom{z}{x} \lambda_1^x \lambda_2^{z-x} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^z}{z!} \end{aligned}$$

So $Z \sim \text{Pois}(\lambda_1 + \lambda_2)$.

3. In a group of 90 people, find an approximation for the probability that there is at least one pair of people such that *they share a birthday and their biological mothers share a birthday* (Assume that no one among the 90 people is the biological mother of another one of the 90 people, nor do two of the 90 people have the same biological mother). You can use $365 \approx 360$, $89 \approx 90$, and $e^x \approx 1 + x$ for $|x|$ small. Verify your approximation result in R: `pbirthday(90, classes=365^2)`.

   Let X be the number of pairs of people for which this coincidence happens. Let I_j be the indicator r.v. for pair j, $j \in \{1, 2, \ldots, \binom{90}{2}\}$. By symmetry, linearity, and the fundamental bridge, we have

$$\mathbb{E}[X] = \sum_{j=1}^{\binom{90}{2}} P(I_j = 1) = \binom{90}{2} \cdot \frac{1}{365^2} \approx \frac{90^2}{2 \cdot 360^2} = \frac{1}{32}$$

   A Poisson approximation for X makes sense since for each pair this coincidence is very unlikely, but there are a lot of pairs, and the indicator r.v.s are approximately independent. So X is approximately $\text{Pois}(\lambda)$, where $\lambda = \mathbb{E}[X] = \frac{1}{32}$ by the property of Poisson. Then we have

$$P(X \geq 1) = 1 - P(X=0) \approx 1 - e^{-\lambda} \approx 1 - (1-\lambda) = \lambda \approx \frac{1}{32}$$

4. Vilfredo would like to study a distribution where the support is (a, ∞) and the PDF is c/x^{b+1} for $x > a$, where a, b, c are constants with $a > 0$ and $b > 2$.

   (a) Find c (in terms of a and b).

   (b) Find the mean (in terms of a and b) of a random variable whose PDF is f.

(c) Vilfredo wants to study this distribution via *simulation*. So he would like to generate i.i.d. draws $X_1, X_2, \dots, X_n$, for some large value of n, with each X_j having PDF f. But he only has access to i.i.d. Unif(0, 1) random variables $U_1, U_2, \dots, U_n$ (by flipping n fair coins, for example). Give a precise, easy-to-implement description of a procedure that he can use to achieve his goal.

(a) By definition of PDF, we have

$$\int_a^\infty \frac{c}{x^{b+1}}\,dx = 1$$

Therefore,

$$\frac{c}{b}a^{-b} = 1$$

We have $c = ba^b$.

(b) By definition of expectation, we have

$$\int_a^\infty \frac{ba^b}{x^b}\,dx = \frac{ba^b}{b-1}a^{1-b} = \frac{ab}{b-1}$$

(c) We know that

$$\int_a^k \frac{c}{x^{b+1}}\,dx = ba^b\left(\frac{1}{b}a^{-b} - \frac{1}{b}k^{-b}\right) = 1 - \left(\frac{a}{k}\right)^b$$

The CDF of X_j is

$$F(x) = \begin{cases} 0, & x \le a \\ 1 - \left(\frac{a}{x}\right)^b, & x > a \end{cases}$$

By universality of uniform, Vilfredo can use the i.i.d. draws $U_1, U_2, \dots, U_n$ to generate

$$X_j = F^{-1}(U_j) = a(1-U_j)^{-\frac{1}{b}}$$