Gov 2001 Section 4, Feb. 28, 2025

Normal Distribution

1. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then the probability density function (PDF) of X is:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

2. If $Z \sim \mathcal{N}(0, 1)$, then the PDF and CDF of Z are:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad \Phi(z) = P(Z \leq z)$$

   You can directly use ϕ and Φ to refer to standard normal PDF and CDF in Psets or exams.

3. For a normal random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ and a standard normal random variable $Z \sim \mathcal{N}(0, 1)$, we use the transformation (you can refer to textbook p.369 change of variables):

$$Z = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1), \quad X = \mu + \sigma Z \sim \mathcal{N}(\mu, \sigma^2)$$

   Thus, we can express probabilities in terms of the standard normal:

$$P(X \leq x) = P(\mu + \sigma Z \leq x) = P\left(Z \leq \frac{x - \mu}{\sigma}\right) = \Phi\left(\frac{x - \mu}{\sigma}\right).$$

4. The normal distribution is symmetric about the mean μ:

$$\phi(y) = \phi(-y), \quad \Phi(y) = 1 - \Phi(-y).$$

5. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then:

$$P(|X - \mu| \leq \sigma) \approx 0.68$$
$$P(|X - \mu| \leq 2\sigma) \approx 0.95$$
$$P(|X - \mu| \leq 3\sigma) \approx 0.997$$

   95% of the values lie within two standard deviations. For standard normal $X \sim \mathcal{N}(0, 1)$, we have $P(X < 1.96) \approx 0.975$ and $P(-1.96 < X < 1.96) = 0.95$.

6. The sum of independent normal r.v.s is also normal. If $X_j \sim \mathcal{N}(\mu_j, \sigma_j^2)$ independently for all $j \in \{1, 2, \ldots, n\}$, then $\sum_{i=1}^n c_i X_i \sim \mathcal{N}\left(\sum_{i=1}^n c_i \mu_i, \sum_{i=1}^n c_i^2 \sigma_i^2\right)$. Proof is not required.

Joint Distributions

1. Joint CDF and PDF:

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$$
$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y)$$

Both the joint PMF and joint PDF must be non-negative and sum/integrate to 1:

$$\sum_x \sum_y P(X=x, Y=y) = 1, \quad \text{(for discrete r.v.s)}$$

$$\int_x \int_y f_{X,Y}(x,y)\, dx\, dy = 1, \quad \text{(for continuous r.v.s)}$$

2. By LOTP, we can calculate the marginal PMF:

$$P(X=x) = \sum_y P(X=x, Y=y) = \sum_y P(X=x|Y=y)P(Y=y)$$

Analogously for marginal PDF:

$$f_X(x) = \int_y f_{X,Y}(x,y)\, dy$$

3. Bayes' rule: For discrete random variables:

$$P(Y=y|X=x) = \frac{P(X=x, Y=y)}{P(X=x)} = \frac{P(X=x|Y=y)P(Y=y)}{P(X=x)}$$

For continuous random variables:

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{f_{X|Y}(x|y)f_Y(y)}{f_X(x)}$$

Hybrid case:

$$f(x|A=a) = \frac{P(A=a|X=x)f(x)}{P(A=a)}$$

4. For discrete/continuous random variables X and Y, independence is equivalent to:

$$P(X \le x, Y \le y) = P(X \le x)P(Y \le y), \quad \forall x, y.$$

For discrete random variables X and Y, independence is equivalent to:

$$P(X=x, Y=y) = P(X=x)P(Y=y), \text{ or } P(X=x \mid Y=y) = P(X=x) \quad \forall x, y.$$

For continuous random variables, independence is equivalent to:

$$f_{X,Y}(x,y) = f_X(x)f_Y(y), \text{ or } f_{X|Y}(x|y) = f_X(x) \quad \forall x, y.$$

5. Multivariate LOTUS (discrete and continuous):

$$E[g(X,Y)] = \sum_x \sum_y g(x,y)P(X=x, Y=y)$$

$$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

6. The covariance between random variables X and Y is:

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y].$$

For any random variables X, Y, Z, W:

- $\text{Cov}(X, X) = \text{Var}(X)$
- $\text{Cov}(X, Y) = \text{Cov}(Y, X)$
- $\text{Cov}(X, c) = 0$ for any constant c
- $\text{Cov}(aX + c, bY + d) = ab\text{Cov}(X, Y)$ for constants a, b, c, d
- $\text{Cov}(X + Y, Z + W) = \text{Cov}(X, Z) + \text{Cov}(X, W) + \text{Cov}(Y, Z) + \text{Cov}(Y, W)$
- $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$

7. The correlation between X and Y is:

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{\text{Cov}(X, Y)}{SD(X)SD(Y)}$$

Correlation is always between -1 and 1. Shifting and scaling does not affect correlation:

$$\text{Corr}(aX + b, Y) = \begin{cases} \text{Corr}(X, Y), & \text{if } a > 0 \\ -\text{Corr}(X, Y), & \text{if } a < 0 \end{cases}$$

And we have $|\text{Corr}(aX + b, X)| = 1$ for any constants a, b.

Multivariate Normal Distribution (MVN)

1. Definition: A vector of random variables $\mathbf{X} = (X_1, X_2, \ldots, X_k)^\top$ follows an MVN if and only if any linear combination of its components is normally distributed. That is,

$$t_1X_1 + t_2X_2 + \cdots + t_kX_k \sim \mathcal{N}, \text{ for any constants } t_1, t_2, \ldots, t_k.$$

The parameters of the MVN distribution are:

- The $k \times 1$ mean vector: $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_k)^\top$
- The $k \times k$ covariance matrix $\mathbf{\Sigma}$, where the (i, j) entry is given by: $\Sigma_{ij} = \text{Cov}(X_i, X_j)$

where each X_i is marginally normal: $X_i \sim \mathcal{N}(\mu_i, \Sigma_{ii})$.

2. If $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{\Sigma})$, then:

   (a) Any subvector (e.g., $(X_2, X_3, \ldots, X_k)^\top$) is also MVN.

   (b) If any two components of an MVN vector are uncorrelated (have zero covariance), then they are independent. This is generally **not true** for arbitrary distributions, but it holds for the MVN (proof not required).

3. The PDF of an MVN vector $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\mathbf{X} = (X_1, X_2, \ldots, X_k)^\top$ is:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{k/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

4. If $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and we define a linear transformation $\mathbf{Y} = A\mathbf{X} + \mathbf{b}$ where $\mathbf{A}$ is an $m \times k$ matrix and $\mathbf{b}$ is an $m \times 1$ constant vector, then $\mathbf{Y}$ also follows an MVN (proof by MGF, not required):

$$\mathbf{Y} \sim \mathcal{N}(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top).$$

Exercise Questions

1. The joint density function of X and Y is given by:

$$f(x, y) = x + y, \quad 0 < x < 1, \quad 0 < y < 1.$$

(a) Check if this is a valid PDF.

(b) Are X and Y independent?

(c) Find $P(X + Y > 3/2)$.

(d) Find the conditional density $f(x|y)$.

(e) Set up an integral to determine the expected value $\mathbb{E}(XY)$.

(a) To be a valid PDF, the total probability must integrate to 1:

$$\int_0^1 \int_0^1 (x + y)\, dy\, dx = 1.$$

Computing the inner integral:

$$\int_0^1 (x + y)\, dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2}.$$

Now integrating with respect to x:

$$\int_0^1 \left(x + \frac{1}{2}\right) dx = \left[\frac{x^2}{2} + \frac{x}{2}\right]_0^1 = \frac{1}{2} + \frac{1}{2} = 1.$$

(b) To check independence, we must verify whether:

$$f(x, y) = f_X(x) f_Y(y).$$

First, we compute the marginal PDFs:

$$f_X(x) = \int_0^1 f(x, y) dy = \int_0^1 (x + y) dy = x + \frac{1}{2}.$$

Similarly, $f_Y(y) = y + \frac{1}{2}$. Since $f(x, y) \neq f_X(x) f_Y(y)$, X and Y are not independent.

(c) If $X+Y>3/2$, X must be at least $1/2$ since Y can't be greater than 1. Rewriting the probability:

$$P(X+Y>3/2)=\int_{1/2}^{1}\int_{3/2-x}^{1}(x+y)dydx.$$

Computing the inner integral:

$$\int_{3/2-x}^{1}(x+y)dy=\left[xy+\frac{y^2}{2}\right]_{3/2-x}^{1}=\frac{1}{2}x^2+x-\frac{5}{8}$$

Evaluating and integrating with respect to x leads to:

$$\int_{1/2}^{1}\left(\frac{1}{2}x^2+x-\frac{5}{8}\right)dx=\frac{5}{24}.$$

(d) By definition,

$$f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}=\frac{x+y}{y+1/2}.$$

(e) Using LOTUS:

$$E(XY)=\int_0^1\int_0^1 xy(x+y)dydx=\int_0^1\left(\frac{x^2}{2}+\frac{x}{3}\right)dx=\frac{1}{3}$$

2. Let $X,Y\overset{\text{i.i.d.}}{\sim}\mathcal{N}(0,1)$.

(a) Find $\text{Cov}(X+Y,X-Y)$.

(b) Prove $X+Y\perp X-Y$.

(a) Using the definition of covariance:

$$\text{Cov}(X+Y,X-Y)=E[(X+Y)(X-Y)]-E(X+Y)E(X-Y).$$

Since the means are zero:

$$E[(X+Y)(X-Y)]=E[X^2-Y^2]=E[X^2]-E[Y^2]=1-1=0.$$

Thus,

$$\text{Cov}(X+Y,X-Y)=0.$$

(b) The properties of normal distribution tell us that any linear combination of X and Y will be normal. Notice that any linear combination of $X+Y$ and $X-Y$ can be written as a linear combination of X and Y. Therefore, by the definition of MVN, $(X+Y,X-Y)$ is an MVN. By MVN property, zero covariance means independence.

3. Let (X,Y) be MVN, while marginally $X,Y\sim\mathcal{N}(0,1)$, and $\text{Corr(X,Y)}=\rho$.

(a) Find the distribution of $X-3Y$.

(b) Find constant c such that $X-3Y$ is independent of $X+cY$.

(a) Because $X - 3Y$ is a linear combination of an MVN, we know that it is normally distributed. We also know

$$\mathbb{E}[X - 3Y] = 0 - 3 \times 0 = 0$$
$$\text{Var}(X - 3Y) = \text{Var}(X) + 9\text{Var}(Y) - 2\text{Cov}(X, 3Y) = 10 - 6\rho$$

(b) We know that any linear combination of $X - 3Y$ and $X + cY$ should be normal, given that any linear combination of X and Y is normal by MVN definition. Therefore, $(X - 3Y, X + cY)$ is also MVN. Therefore, by MVN property, we only need

$$\begin{aligned}\text{Cov}(X - 3Y, X + cY) &= \text{Var}(X) + (c - 3)\text{Cov}(X, Y) - 3c\text{Var}(Y) \\ &= 1 + (c - 3)\rho - 3c \\ &= 1 - 3\rho - (3 - \rho)c = 0\end{aligned}$$

Solve the equation, we get $c = \frac{1-3\rho}{3-\rho}$.