13: More on Ordinary Least Square

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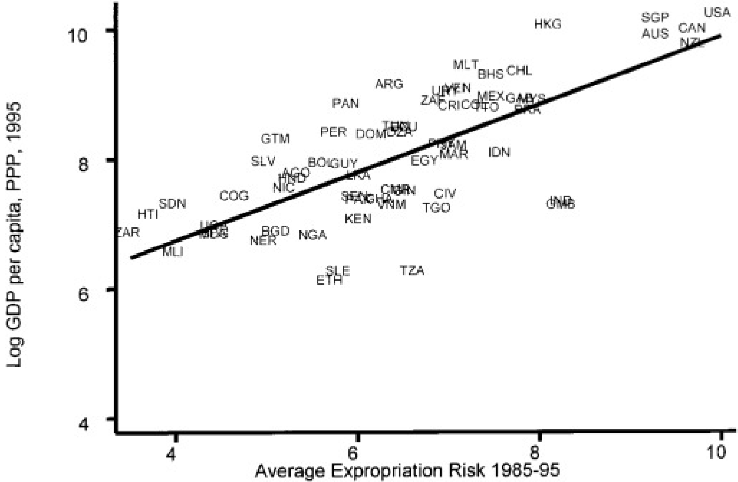
Where are we? Where are we going?

- Before: learned about CEFs and linear projections in the population.
- Last time: OLS estimator, its algebraic properties.
- Now: its statistical properties, both finite-sample and asymptotic.

Acemoglu, Johnson, Robinson 2001



Sampling distribution of the OLS estimator

- OLS is an estimator—we plug data into and we get out estimates.

Sample 1: $\{(Y_1, X_1), \ldots, (Y_n, X_n)\}$

Sample 2: $\{(Y_1, X_1), \ldots, (Y_n, X_n)\}$

$\vdots$

Sample $k-1$: $\{(Y_1, X_1), \ldots, (Y_n, X_n)\}$

Sample k: $\{(Y_1, X_1), \ldots, (Y_n, X_n)\}$

OLS

$(\hat{\beta}_0, \hat{\beta}_1)_1$

$(\hat{\beta}_0, \hat{\beta}_1)_2$

$\vdots$

$(\hat{\beta}_0, \hat{\beta}_1)_{k-1}$

$(\hat{\beta}_0, \hat{\beta}_1)_k$

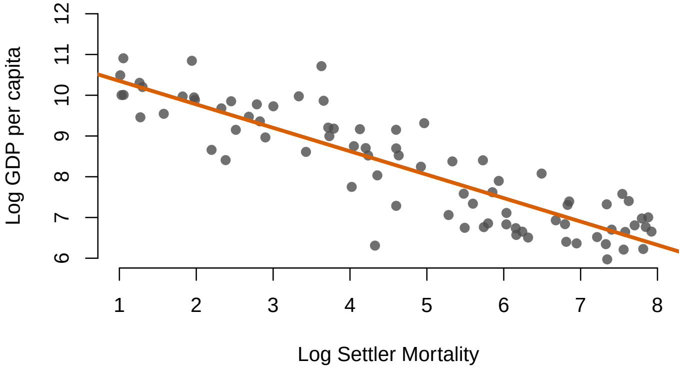
- Just like the sample mean or sample difference in means
- Has a sampling distribution, with a sampling variance/standard error.

Simulation procedure

- Let's take a simulation approach to demonstrate:
  - Pretend that our data represents the population of interest
  - See how the line varies from sample to sample

1. Draw a random sample of size $n = 30$ with replacement using `sample()`
2. Use `lm()` to calculate the OLS estimates of the slope and intercept
3. Plot the estimated regression line

Population Data



Sample Data

Estimated Regression Line from Sample
Log GDP per capita
7
8
9
10
11
1
2
3
4
5
6
7
8
Log Settler Mortality

Big picture

- We want finite-sample guarantees about our estimates.
  - Unbiasedness, exact sampling distribution, etc.
- But finite-sample results come at a price in terms of assumptions.
  - Unbiasedness: CEF is linear.
  - Exact sampling distribution: normal errors.
- Asymptotic results hold under much weaker assumptions, but require more data.
  - OLS consistent for the linear projection even with nonlinear CEF.
  - Asymptotic normality for sampling distribution under mild assumptions.
- Focus on two models.
  - **Linear projection model** for asymptotic results.
  - **Linear regression/CEF model** for finite samples.

Linear projection model

- We'll start at the most broad, fewest assumptions

Linear projection model

1. For the variables $(Y, \mathbf{X})$, we assume the linear projection of Y on $\mathbf{X}$ is defined as:
$$Y = \mathbf{X}'\beta + e$$
$$\mathbb{E}[\mathbf{X}e] = 0.$$
2. The design matrix is invertible, so $\mathbb{E}[\mathbf{X}_i\mathbf{X}_i'] > 0$ (positive definite).

- Linear projection model holds under **very** mild assumptions.
  - Remember: not even assuming linear CEF!
  - Implies coefficients are $\beta = (\mathbb{E}[\mathbf{X}\mathbf{X}'])^{-1}\mathbb{E}[\mathbf{X}Y]$
- What properties can we derive under such weak assumptions?

A very useful decomposition

$$\hat{\beta} = \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i\mathbf{x}_i'\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i Y_i\right) = \beta + \underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i\mathbf{x}_i'\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i e_i\right)}_{\text{estimation error}}$$

- OLS estimates are the truth plus some estimation error.
- Most of what we derive about OLS comes from this view.
- Sample means in the estimation error follow the law of large numbers:

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i\mathbf{x}_i' \xrightarrow{p} \mathbb{E}[\mathbf{X}_i\mathbf{X}_i'] \equiv \mathbf{Q}_{xx} \qquad \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i e_i \xrightarrow{p} \mathbb{E}[\mathbf{X}e] = 0$$

- $\mathbf{Q}_{xx}$ is invertible by assumption, so by the continuous mapping theorem:

$$\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i\mathbf{x}_i'\right)^{-1} \xrightarrow{p} \mathbf{Q}_{xx}^{-1} \quad \Rightarrow \quad \hat{\beta} \xrightarrow{p} \beta + \mathbf{Q}_{xx}^{-1} \cdot 0 = \beta,$$

Consistency of OLS

Theorem (Consistency of OLS)

Under the linear projection model and i.i.d. data, $\hat{\beta}$ is consistent for β.

- Simple proof, but powerful result.
- OLS consistently estimates the linear projection coefficients, β.
  - No guarantees about what the β_j represent!
  - Best linear approximation to $\mathbb{E}[Y \mid \mathbf{X}]$.
  - If we have a linear CEF, then it's consistent for the CEF coefficients.
- Valid with no restrictions on Y: could be binary, discrete, etc.
- Not guaranteed to be unbiased (unless CEF is linear, as we'll see...)

Central limit theorem, reminders

- We'll want to approximate the sampling distribution of $\hat{\beta}$. CLT!
- Consider some sample mean of i.i.d. data: $n^{-1}\sum_{i=1}^{n} g(\mathbf{X}_i)$. We have:

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} g(\mathbf{X}_i)\right] = \mathbb{E}[g(\mathbf{X}_i)] \qquad \text{var}\left[\frac{1}{n}\sum_{i=1}^{n} g(\mathbf{X}_i)\right] = \frac{\text{var}[g(\mathbf{X}_i)]}{n}$$

- CLT implies:

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} g(\mathbf{X}_i) - \mathbb{E}[g(\mathbf{X}_i)]\right) \xrightarrow{d} \mathcal{N}(0, \text{var}[g(\mathbf{X}_i)])$$

- If $\mathbb{E}[g(\mathbf{X}_i)] = 0$, then we have:

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} g(\mathbf{X}_i)\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} g(\mathbf{X}_i) \xrightarrow{d} \mathcal{N}\left(0, \mathbb{E}[g(\mathbf{X}_i)g(\mathbf{X}_i)']\right)$$

Standardized estimator

$$\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i' \right)^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{x}_i e_i \right)$$

- Remember that $\left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i' \right)^{-1} \xrightarrow{p} \mathbf{Q}_{xx}^{-1}$ so we have

$$\sqrt{n}(\hat{\beta} - \beta) \approx \mathbf{Q}_{xx}^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{x}_i e_i \right)$$

- What about $n^{-1/2} \sum_{i=1}^{n} \mathbf{x}_i e_i$? Notice that:
  - $n^{-1} \sum_{i=1}^{n} \mathbf{x}_i e_i$ is a sample average with $\mathbb{E}[\mathbf{X}_i e_i] = 0$.
  - Rewrite as $\sqrt{n}$ times an average of i.i.d. mean-zero random vectors.
- Let $\Omega = \mathbb{E}[e_i^2 \mathbf{x}_i \mathbf{x}_i']$ and apply the CLT:

$$\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{x}_i e_i \right) \xrightarrow{d} \mathcal{N}(0, \Omega)$$

Asymptotic normality

Theorem (Asymptotic Normality of OLS)

Under the linear projection model,

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} \mathcal{N}(0, \mathbf{V}_\beta),$$

where,

$$\mathbf{V}_\beta = \mathbf{Q}_{xx}^{-1}\Omega\mathbf{Q}_{xx}^{-1} = \left(\mathbb{E}[\mathbf{X}_i\mathbf{X}_i']\right)^{-1}\mathbb{E}[e_i^2\mathbf{x}_i\mathbf{x}_i']\left(\mathbb{E}[\mathbf{X}_i\mathbf{X}_i']\right)^{-1}$$

- $\hat{\beta}$ is approximately normal with mean β and variance $\mathbf{V}_{\hat{\beta}} = \mathbf{Q}_{xx}^{-1}\Omega\mathbf{Q}_{xx}^{-1}/n$
- $\mathbf{V}_{\hat{\beta}} = \mathbf{V}_\beta/n$ is the **asymptotic covariance matrix** of $\hat{\beta}$
  - Square root of the diagonal of $\mathbf{V}_{\hat{\beta}}$ = standard errors for $\hat{\beta}_j$
- Allows us to formulate (approximate) confidence intervals, tests.

Estimating OLS variance

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} \mathcal{N}(0, \mathbf{V}_{\beta}), \qquad \mathbf{V}_{\beta} = \mathbf{Q}_{xx}^{-1}\Omega\mathbf{Q}_{xx}^{-1}$$

- Estimation of $\mathbf{V}_{\beta}$ uses plug-in estimators.
  - Replace $\mathbf{Q}_{xx} = \mathbb{E}[\mathbf{X}_i\mathbf{X}_i']$ with $\hat{\mathbf{Q}}_{xx} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i\mathbf{x}_i' = \mathbf{X}'\mathbf{X}/n$.
  - Replace $\Omega = \mathbb{E}[e_i^2\mathbf{x}_i\mathbf{x}_i']$ with $\hat{\Omega} = \frac{1}{n}\sum_{i=1}^{n} \hat{e}_i^2\mathbf{x}_i\mathbf{x}_i'$.
- Putting these together to get a **consistent** estimator:

$$\hat{\mathbf{V}}_{\beta} = \left(\frac{1}{n}\mathbf{X}'\mathbf{X}\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} \hat{e}_i^2\mathbf{x}_i\mathbf{x}_i'\right) \left(\frac{1}{n}\mathbf{X}'\mathbf{X}\right)^{-1} \xrightarrow{p} \mathbf{V}_{\beta}$$

- Approximate variance of the coefficients:

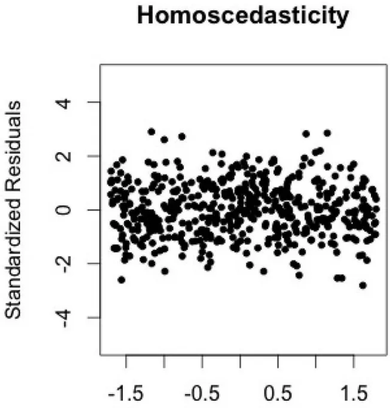
$$\hat{\mathbf{V}}_{\hat{\beta}} = \frac{1}{n}\hat{\mathbf{V}}_{\beta} = (\mathbf{X}'\mathbf{X})^{-1} \left(\sum_{i=1}^{n} \hat{e}_i^2\mathbf{x}_i\mathbf{x}_i'\right) (\mathbf{X}'\mathbf{X})^{-1}$$

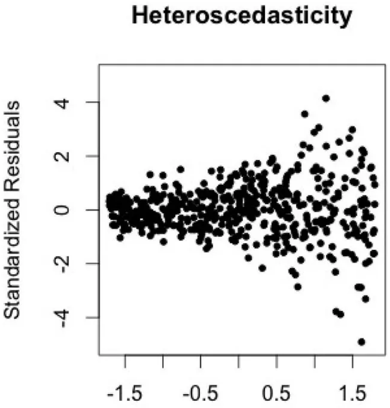
- Square root of the diagonal of $\hat{\mathbf{V}}_{\hat{\beta}}$:
  heteroskedasticity-consistent (HC) SEs (aka “robust SEs”)

Homoskedasticity

Assumption: Homoskedasticity

The variance of the error terms is constant in $\mathbf{X}$,
$\mathbb{E}[e^2 \mid \mathbf{X}] = \sigma^2(\mathbf{X}) = \sigma^2$.





Consequences of homoskedasticity

- Homoskedasticity implies $\mathbb{E}[e_i^2 \mathbf{X}_i \mathbf{X}_i'] = \mathbb{E}[e_i^2]\mathbb{E}[\mathbf{X}_i \mathbf{X}_i'] = \sigma^2 \mathbf{Q}_{xx}$
- Simplifies the expression for the variance of $\sqrt{n}(\hat{\beta} - \beta)$:

$$\mathbf{V}_\beta^{\mathsf{lm}} = \mathbf{Q}_{xx}^{-1}\mathbb{E}[e_i^2]\mathbf{Q}_{xx}\mathbf{Q}_{xx}^{-1} = \sigma^2 \mathbf{Q}_{xx}^{-1}$$

- Estimated variance of $\hat{\beta}$ under homoskedasticity

$$s^2 = \frac{1}{n-k}\sum_{i=1}^{n} \hat{e}_i^2 \qquad \hat{\mathbf{V}}_\beta^{\mathsf{lm}} = \frac{1}{n}s^2\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i'\right)^{-1} = s^2(\mathbf{X}'\mathbf{X})^{-1}$$

- LLN implies $s^2 \xrightarrow{p} \sigma^2$ and so $n\hat{\mathbf{V}}_\beta^{\mathsf{lm}}$ is consistent for $\mathbf{V}_\beta^{\mathsf{lm}}$

Notes on skedasticity

- Homoskedasticity: strong assumption that isn't needed for consistency.
- Software: almost always reports $\hat{\mathbf{V}}_{\beta}^{\text{lm}}$ by default.
  - e.g. `lm()` in R or `reg` in Stata.
- Separate commands for HC SEs $\hat{\mathbf{V}}_{\beta}$
  - Use `{sandwich}` package in R or `, robust` in Stata.
- If $\hat{\mathbf{V}}_{\beta}^{\text{lm}}$ and $\hat{\mathbf{V}}_{\beta}$ differ a lot, maybe check modeling assumptions (King and Roberts, PA 2015).
- Lots of "flavors" of HC variance estimators (HC0, HC1, HC2, etc).
  - Mostly small, ad hoc changes to improve finite-sample performance.

AJR Data

```r
library(sandwich)
mod <- lm(logpgp95 ~ avexpr + lat_abst + meantemp, data = ajr)
vcov(mod) ## homoskdastic V_\hat{beta}
```

```
##             (Intercept)    avexpr  lat_abst  meantemp
## (Intercept)      0.9079 -0.040952 -0.537463 -0.023246
## avexpr          -0.0410  0.004162 -0.000778  0.000605
## lat_abst        -0.5375 -0.000778  0.867588  0.016717
## meantemp        -0.0232  0.000605  0.016717  0.000705
```

```r
sandwich::vcovHC(mod, type = "HC2") ## HC2
```

```
##             (Intercept)   avexpr lat_abst  meantemp
## (Intercept)      0.9764 -0.05735 -0.29548 -0.024639
## avexpr          -0.0573  0.00538 -0.00358  0.001107
## lat_abst        -0.2955 -0.00358  0.60821  0.008792
## meantemp        -0.0246  0.00111  0.00879  0.000706
```

Inference with OLS

- Inference is basically the same as any asymptotically normal estimator.
- Let $\widehat{\text{se}}(\hat{\beta}_j)$ be the estimated SE for $\hat{\beta}_j$.
  - Square root of jth diagonal entry: $\sqrt{[\hat{\mathbf{V}}_{\hat{\beta}}]_{jj}}$
- Hypothesis test of $\beta_j = b_0$:

$$\text{general t-statistic} = \frac{\hat{\beta}_j - b_0}{\widehat{\text{se}}(\hat{\beta}_j)} \qquad \text{``usual'' t-statistic} = \frac{\hat{\beta}_j}{\widehat{\text{se}}(\hat{\beta}_j)}$$

  Use same critical values from the normal as usual $z_{\alpha/2} = 1.96$.

- 95% (asymptotic) confidence interval for $\hat{\beta}_j$:

$$\left[\hat{\beta}_j - 1.96\widehat{\text{se}}(\hat{\beta}_j), \quad \hat{\beta}_j + 1.96\widehat{\text{se}}(\hat{\beta}_j)\right]$$

- Software often uses t critical values instead of normal (we'll see why).

lmtest and coeftest

```r
library(lmtest)
## homoskedastic error
lmtest::coeftest(mod)
```

```
##
## t test of coefficients:
##
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   6.9289     0.9528    7.27  1.2e-09 ***
## avexpr        0.4059     0.0645    6.29  5.1e-08 ***
## lat_abst     -0.1980     0.9314   -0.21    0.832
## meantemp     -0.0641     0.0266   -2.41    0.019 *
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```r
## HC2 variance estimator
lmtest::coeftest(mod, vcov = vcovHC(mod, type = "HC2"))
```

```
##
## t test of coefficients:
##
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   6.9289     0.9881    7.01  3.3e-09 ***
## avexpr        0.4059     0.0733    5.53  8.6e-07 ***
## lat_abst     -0.1980     0.7799   -0.25    0.801
## meantemp     -0.0641     0.0266   -2.41    0.019 *
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Inference for interactions

$$m(x, z) = \beta_0 + X\beta_1 + Z\beta_2 + XZ\beta_3$$

- **Partial** or **marginal** effect of X at Z: $\frac{\partial m(x,z)}{\partial x} = \beta_1 + z\beta_3$
- Estimate it by plugging in the estimated coefficients: $\frac{\partial \hat{m}(x,z)}{\partial x} = \hat{\beta}_1 + z\hat{\beta}_3$
- What if we want the variance of this effect for any value of Z?

$$\mathbb{V}\left(\frac{\partial \hat{m}(x, z)}{\partial x}\right) = \mathbb{V}\left[\hat{\beta}_1 + z\hat{\beta}_3\right] = \mathbb{V}[\hat{\beta}_1] + z^2\mathbb{V}[\hat{\beta}_3] + 2z\,\mathrm{cov}[\hat{\beta}_1, \hat{\beta}_3]$$

- Use the estimated covariance matrix:

$$\widehat{\mathbb{V}}\left(\frac{\partial \hat{m}(x, z)}{\partial x}\right) = \hat{\mathbf{V}}_{\hat{\beta}_1} + z^2\hat{\mathbf{V}}_{\hat{\beta}_3} + 2z\hat{\mathbf{V}}_{\hat{\beta}_1,\hat{\beta}_3}$$

- $\hat{\mathbf{V}}_{\hat{\beta}_1}$ is the diagonal entry of $\hat{\mathbf{V}}_{\hat{\beta}}$ for $\hat{\beta}_1$

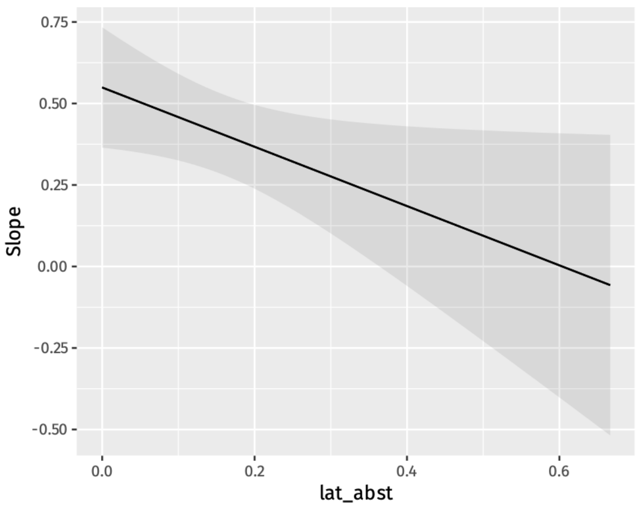
Visualizing

```r
int_mod <- lm(logpgp95 ~ avexpr * lat_abst + meantemp, data = ajr)
coeftest(int_mod)
```

```
##
## t test of coefficients:
##
##                 Estimate Std. Error t value Pr(>|t|)
## (Intercept)       6.9864     0.9273    7.53    5e-10
## avexpr            0.5491     0.0941    5.84    3e-07
## lat_abst          5.8152     3.0791    1.89   0.0642
## meantemp         -0.1048     0.0326   -3.21   0.0022
## avexpr:lat_abst  -0.9095     0.4451   -2.04   0.0458
##
## (Intercept)     ***
## avexpr          ***
## lat_abst        .
## meantemp        **
## avexpr:lat_abst *
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Visualizing

```r
library(marginaleffects)
plot_slopes(int_mod, variables = "avexpr", condition = "lat_abst")
```



Tests of multiple coefficients

$$m(X, Z) = \beta_0 + X\beta_1 + Z\beta_2 + XZ\beta_3$$

- What about a test of no effect of X ever? Involves 2 coefficients:

$$H_0 : \beta_1 = \beta_3 = 0$$

- Alternative: $H_1 : \beta_1 \neq 0$ or $\beta_3 \neq 0$
- We would like a test statistic that is large when the null is implausible.
    - What about $\hat{\beta}_1^2 + \hat{\beta}_3^2$?
    - Distribution depends on the variance/covariance of the coefficients.
    - Need to normalize like the t-statistic.

Alternative test for one coefficient

- Usually t-test of $H_0 : \beta_j = b_0$ based on the t-statistic:

$$t = \frac{\hat{\beta}_j - b_0}{\widehat{\mathsf{se}}(\hat{\beta}_j)},$$

- Reject when $|t| > c$ for some critical value c from the standard normal.
- Equivalent test based rejects when $t^2 > c^2$

$$t^2 = \frac{(\hat{\beta}_j - b_0)^2}{\mathbb{V}[\hat{\beta}_j]} = \frac{n(\hat{\beta}_j - b_0)^2}{[\hat{\mathbf{V}}_{\hat{\beta}}]_{jj}}$$

- Because $t \xrightarrow{d} \mathcal{N}(0, 1)$, we'll have t^2 converging to a χ^2_1 distribution
  - Reminder: χ^2_k is the sum of k squared standard normals.
  - Could get the critical value for t^2 directly from χ^2_1.

Rewriting hypotheses with matrices

- We can rewrite the null hypothesis as $H_0 : \mathbf{L}\beta = \mathbf{c}$ where,

$$\mathbf{L} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  - $\mathbf{L}$ has q rows or restriction and $k+1$ columns (one for each coefficient)

- Estimated version of the constraint: $\mathbf{L}\hat{\beta}$
- By the Delta method, under the null hypothesis we have

$$\sqrt{n}\left(\mathbf{L}\hat{\beta} - \mathbf{L}\beta\right) \xrightarrow{d} \mathcal{N}(0, \mathbf{L}\mathbf{V}_\beta\mathbf{L}')$$

- In this case:

$$\sqrt{n}\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_3 \end{pmatrix} \xrightarrow{d} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} [\mathbf{V}_\beta]_{11} & [\mathbf{V}_\beta]_{13} \\ [\mathbf{V}_\beta]_{31} & [\mathbf{V}_\beta]_{33} \end{pmatrix}\right)$$

- If this covariance matrix were identity, then these would be standard normal and $\hat{\beta}_1^2 + \hat{\beta}_3^2$ would be χ^2_2 under the null

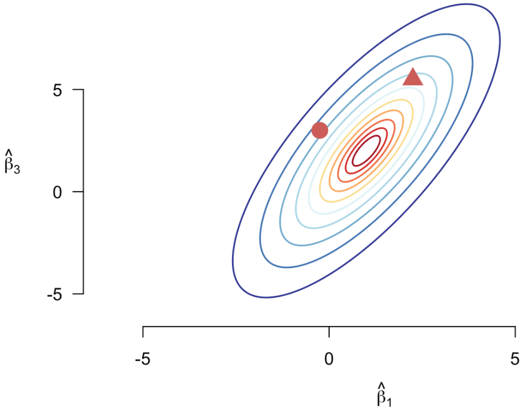
Wald statistic

- Under the null, $\sqrt{n}(\mathbf{L}\hat{\beta} - \mathbf{c}) \xrightarrow{d} \mathcal{N}(0, \mathbf{L}\mathbf{V}_\beta\mathbf{L}')$
- $(\mathbf{L}\hat{\beta} - \mathbf{c})'(\mathbf{L}\hat{\beta} - \mathbf{c})$ is the squared deviations from the null.
  - Problem: doesn't account for variance/covariance of the estimated coefficients.
- **Wald statistic** normalize by the covariance matrix:

$$W = n(\mathbf{L}\hat{\beta} - \mathbf{c})' \left(\mathbf{L}\hat{\mathbf{V}}_\beta\mathbf{L}'\right)^{-1} (\mathbf{L}\hat{\beta} - \mathbf{c})$$

- Similar to dividing by the SE for the t-test
- Squared distance of observed values from the null, weighted by the distribution of the parameters under the null

Weighting by the distribution



Wald test

$$W = n(\mathbf{L}\hat{\beta} - \mathbf{c})' \left(\mathbf{L}\hat{\mathbf{V}}_{\beta}\mathbf{L}'\right)^{-1} (\mathbf{L}\hat{\beta} - \mathbf{c})$$

- Asymptotically under the null $W \xrightarrow{d} \chi^2_q$ where q is rows of $\mathbf{L}$
  - q is the number of linear restrictions in the null
- **Wald test**: reject when $W > w_\alpha$, where $\mathbb{P}(W > w_\alpha) = \alpha$ under the null.
  - Use χ^2_q distribution for critical values, p-values
- Typical software output: **F-statistic** $F = W/q$
  - p-values and critical values come from F distribution with q and $n - k - 1$ dfs.
  - As $n \to \infty$, $F_{q,n-k-1} \xrightarrow{d} \chi^2_q$ so asymptotically similar to Wald under homoskedasticity (slightly more conservative).
  - No justification for F test under heteroskedasticity.
  - "Usual" F-test reports test of all coef $= 0$ except intercept (pointless?)

Wald test steps

1. Choose a Type I error rate, α.
   - Same interpretation: rate of false positives you are willing to accept
2. Calculate the rejection region for the test (one-sided)
   - Rejection region is the region $W > w_\alpha$ such that $\mathbb{P}(W > w_\alpha) = \alpha$
   - We can get this from R using the `qchisq()` function
3. Reject if observed statistic is bigger than critical value
   - Use `pchisq()` to get p-values if needed.
   - When applied to a single coefficient, equivalent to a t-test.
   - Use packages like `{lmtest}` or `{clubSandwich}` in R.

Wald Test

```r
## run OLS with the restrictions imposed (avexpr removed)
restricted <- lm(logpgp95 ~ lat_abst + meantemp, data = ajr)

## pass estimated model and estimated null model to
## wald test with HC variance estimator
lmtest::waldtest(restricted, int_mod, test = "Chisq",
                 vcov = vcovHC)
```

```
## Wald test
##
## Model 1: logpgp95 ~ lat_abst + meantemp
## Model 2: logpgp95 ~ avexpr * lat_abst + meantemp
##   Res.Df Df Chisq Pr(>Chisq)
## 1     57
## 2     55  2  34.2    3.7e-08 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Multiple testing

- Separate t-tests for each β_j: α of them will be significant by chance.
- Illustration:
    - Randomly draw 21 variables independently.
    - Run a regression of the first variable on the rest.
- By design, no effect of any variable on any other.

Example with Multiple testing

```r
noise <- data.frame(matrix(rnorm(2100), nrow = 100, ncol = 21))
summary(lm(noise))
```

```
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.028039   0.113820   -0.25   0.8061
## X2          -0.150390   0.112181   -1.34   0.1839
## X3           0.079158   0.095028    0.83   0.4074
## X4          -0.071742   0.104579   -0.69   0.4947
## X5           0.172078   0.114002    1.51   0.1352
## X6           0.080852   0.108341    0.75   0.4577
## X7           0.102913   0.114156    0.90   0.3701
## X8          -0.321053   0.120673   -2.66   0.0094 **
## X9          -0.053122   0.107983   -0.49   0.6241
## X10          0.180105   0.126443    1.42   0.1583
## X11          0.166386   0.110947    1.50   0.1377
## X12          0.008011   0.103766    0.08   0.9387
## X13          0.000212   0.103785    0.00   0.9984
## X14         -0.065969   0.112214   -0.59   0.5583
## X15         -0.129654   0.111575   -1.16   0.2487
## X16         -0.054446   0.125140   -0.44   0.6647
## X17          0.004335   0.112012    0.04   0.9692
## X18         -0.080796   0.109853   -0.74   0.4642
## X19         -0.085806   0.118553   -0.72   0.4713
## X20         -0.186006   0.104560   -1.78   0.0791 .
## X21          0.002111   0.108118    0.02   0.9845
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.999 on 79 degrees of freedom
## Multiple R-squared:  0.201,  Adjusted R-squared:  -0.00142
## F-statistic: 0.993 on 20 and 79 DF,  p-value: 0.48
```

Multiple testing gives false positives

- 1 out of 20 variables significant at $\alpha = 0.05$
- 2 out of 20 variables significant at $\alpha = 0.1$
- Exactly the number of false positives we would expect.
- But notice the F-statistic: the variables are not **jointly** significant
- **Bonferroni correction**: use p-value cutoff α/m where m is the number of hypotheses.
  - Example: $0.05/20 = 0.0025$
  - Ensures that the family-wise error rate (probability of making at least 1 Type I error) is less than α.

Standard linear regression model

- Standard textbook model: **correctly specified linear CEF**
  - Designed for finite-sample results.

Assumption: Linear Regression Model

1. The variables $(Y, \mathbf{X})$ satisfy the the linear CEF assumption.

$$Y = \mathbf{X}'\beta + e$$

$$\mathbb{E}[e \mid \mathbf{X}] = 0.$$

2. The design matrix is invertible $\mathbb{E}[\mathbf{X}\mathbf{X}'] > 0$ (positive definite).

- Basically this assumes the CEF of Y given $\mathbf{X}$ is linear.
- We continue to maintain $\{(Y_i, \mathbf{X}_i)\}$ are i.i.d.

Properties of OLS under linear CEF

- Linear CEFs imply stronger finite-sample guarantees:

1. **Unbiasedness**: $\mathbb{E}[\hat{\beta} \mid \mathbf{X}] = \beta$
2. **Conditional sampling variance**: let $\sigma_i^2 = \mathbb{E}[e_i^2 \mid \mathbf{X}_i]$

$$\mathbb{V}[\hat{\beta} \mid \mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1} \left(\sum_{i=1}^{n} \sigma_i^2 \mathbf{x}_i \mathbf{x}_i' \right) (\mathbf{X}'\mathbf{X})^{-1}$$

- Useful when linearity holds by default (discrete $\mathbf{X}$ in experiments, etc)

Linear CEF under homoskedasticity

- Under homoskedasticity, we have a few other finite-sample results:

3. **Conditional sampling variance**: $\mathbb{V}[\hat{\beta} \mid \mathbf{X}] = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$
4. **Unbiased variance estimator**: $\mathbb{E}[\hat{\mathbf{V}}^0(\hat{\beta}) \mid \mathbf{X}] = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$
5. **Gauss-Markov**: OLS is the best linear unbiased estimator of β (BLUE). If $\tilde{\beta}$ is a linear estimator,

$$\mathbb{V}[\tilde{\beta} \mid \mathbf{X}] \geq \mathbb{V}[\hat{\beta} \mid \mathbf{X}] = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$$

- For matrices, $\mathbf{A} \geq \mathbf{B}$ means that $\mathbf{A} - \mathbf{B}$ is positive semidefinite.
- A matrix $\mathbf{C}$ is p.s.d. if $\mathbf{x}'\mathbf{C}\mathbf{x} \geq 0$.
- Upshot: OLS will have the smaller SEs than any other linear estimator.

Normal regression model

- Most parametric: $Y \sim \mathcal{N}(\mathbf{X}'\beta, \sigma^2)$.
  - Normal error model since $e = Y - \mathbf{X}'\beta \sim \mathcal{N}(0, \sigma^2)$.
- Rarely believed, but allows for exact inference for all n.
  - $(\hat{\beta}_j - \beta_j)/\widehat{\text{se}}(\hat{\beta}_j)$ follows a t distribution with $n - k$ degrees of freedom.
  - F statistics follows F distribution exactly rather than approximately.
- Software often implicitly assumes this for p-values.
- With reasonable n, asymptotic normality has the same effect.

Clustered dependence: intuition

- Think back to the Gerber, Green, and Larimer (2008) social pressure mailer example.
  - Randomly assign households to different treatment conditions.
  - But the measurement of turnout is at the individual level.
- Zero conditional mean error holds here (random assignment)
- Violation of **iid/random sampling**:
  - errors of individuals within the same household are correlated.
  - SEs are going to be wrong.
- Called **clustering** or **clustered dependence**

Clustered dependence: notation

- Clusters (groups): $g = 1, \dots, m$
- Units: $i = 1, \dots, n_g$
- n_g is the number of units in cluster g
- $n = \sum_{g=1}^{m} n_g$ is the total number of units
- Units are (usually) belong to a single cluster:
  - voters in households
  - individuals in states
  - students in classes
  - rulings in judges
- Outcome varies at the unit-level, Y_{ig} and the main independent variable varies at the cluster level, X_g.

Clustered dependence: example model

$$Y_{ig} = \beta_0 + X_g\beta_1 + \nu_{ig}$$
$$= \beta_0 + X_g\beta_1 + c_g + u_{ig}$$

- u_{ig} unit error component with $\mathbb{V}[u_{ig} \mid X_g] = \sigma_u^2$
- c_g cluster error component with $\mathbb{V}[c_g \mid X_g] = \sigma_c^2$
- c_g and u_{ig} are assumed to be independent of each other.
  - $\Rightarrow \mathbb{V}[\nu_{ig} \mid X_g] = \sigma_c^2 + \sigma_u^2$
- What if we ignore this structure and just use ν_{ig} as the error?

Lack of independence

- Covariance between two units i and s in the same cluster:

$$\text{Cov}[\nu_{ig}, \nu_{sg}] = \sigma_c^2$$

- Correlation between units in the same group is called the **intra-class correlation coefficient**, or ρ_c:

$$\text{Cor}[\nu_{ig}, \nu_{sg}] = \frac{\sigma_c^2}{\sigma_c^2 + \sigma_u^2} = \rho_c$$

- Zero covariance of two units i and s in different clusters g and k:

$$\text{Cov}[\nu_{ig}, \nu_{sk}] = 0$$

Example covariance matrix

- $\mathbf{v}' = [\nu_{1,1} \quad \nu_{2,1} \quad \nu_{3,1} \quad \nu_{4,2} \quad \nu_{5,2} \quad \nu_{6,2}]$
- Variance matrix under clustering:

$$\mathbb{V}[\mathbf{v} \mid \mathbf{X}] = \begin{bmatrix} \sigma_c^2 + \sigma_u^2 & \sigma_c^2 & \sigma_c^2 & 0 & 0 & 0 \\ \sigma_c^2 & \sigma_c^2 + \sigma_u^2 & \sigma_c^2 & 0 & 0 & 0 \\ \sigma_c^2 & \sigma_c^2 & \sigma_c^2 + \sigma_u^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_c^2 + \sigma_u^2 & \sigma_c^2 & \sigma_c^2 \\ 0 & 0 & 0 & \sigma_c^2 & \sigma_c^2 + \sigma_u^2 & \sigma_c^2 \\ 0 & 0 & 0 & \sigma_c^2 & \sigma_c^2 & \sigma_c^2 + \sigma_u^2 \end{bmatrix}$$

- Variance matrix under i.i.d.:

$$\mathbb{V}[\mathbf{v} \mid \mathbf{X}] = \begin{bmatrix} \sigma_u^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_u^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_u^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_u^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_u^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_u^2 \end{bmatrix}$$

Effects of clustering

$$Y_{ig} = \beta_0 + X_g\beta_1 + c_g + u_{ig}$$

- $\mathbb{V}^0[\hat{\beta}_1]$ = **conventional** OLS variance assuming i.i.d./homoskedasticity.
- Let $\mathbb{V}[\hat{\beta}_1]$ be the true sampling variance under clustering.
- When clusters are balanced, $n^* = n_g$, comparison of clustered to conventional:

$$\mathbb{V}[\hat{\beta}_1] \approx \mathbb{V}^0[\hat{\beta}_1]\,(1 + (n^* - 1)\rho_c)$$

- True variance will be higher than conventional when within-cluster correlation is positive, $\rho_c > 0$.

Linear model with clustering

$$Y_{ig} = \mathbf{X}_{ig}'\beta + \nu_{ig}$$

- Assumptions:
  - $\mathbb{E}[\nu_{ig} \mid \mathbf{X}_{ig}] = 0$ so we have the correct CEF.
  - $\mathbb{E}[\nu_{ig}\nu_{jg'} \mid \mathbf{X}_{ig}, \mathbf{X}_{jg'}] = 0$ unless $g = g'$.
  - Correlated errors allowed within groups, uncorrelated across. Allows heteroskedasticity.
- Pooled OLS under clustered dependence:

$$\mathbf{Y}_g = \mathbf{X}_g\beta + \mathbf{v}_g$$

- $\mathbf{Y}_g$ is the $n_g \times 1$ vector of responses for cluster g
- $\mathbf{X}_g$ is the $n_g \times k$ matrix of data for the gth cluster.
- We can write the OLS estimator as:

$$\hat{\beta} = \left(\sum_{g=1}^{m} \mathbf{X}_g'\mathbf{X}_g\right)^{-1} \left(\sum_{g=1}^{m} \mathbf{X}_g'\mathbf{Y}_g\right)$$

Cluster-robust variance estimator

- Independence is across clusters so the CLT holds as m gets big.
  - Key intuition: we're sampling clusters, not individual units.
- CLT implies $\sqrt{m}(\hat{\beta} - \beta)$ will be asymp. normal with mean 0 and variance:

$$\left(\mathbb{E}[\mathbf{X}_g'\mathbf{X}_g]\right)^{-1} \mathbb{E}[\mathbf{X}_g'\mathbf{v}_g\mathbf{v}_g'\mathbf{X}_g] \left(\mathbb{E}[\mathbf{X}_g'\mathbf{X}_g]\right)^{-1}$$

- Similar to the iid case, replace population quantities with sample versions (and divide by m):

$$\hat{\mathbb{V}}_{\hat{\beta}}^{\mathsf{CL0}} = (\mathbf{X}'\mathbf{X})^{-1} \left(\sum_{g=1}^{m} \mathbf{X}_g'\hat{\mathbf{v}}_g\hat{\mathbf{v}}_g'\mathbf{X}_g\right) (\mathbf{X}'\mathbf{X})^{-1}$$

Noting: $\mathbf{X}'\mathbf{X}/m = m^{-1}\sum_{g=1}^{m} \mathbf{X}_g'\mathbf{X}_g$

- With small-sample adjustment (reported by most software):

$$\hat{\mathbb{V}}_{\hat{\beta}}^{\mathsf{CL1}} = \frac{m}{m-1} \cdot \frac{n-1}{n-k}(\mathbf{X}'\mathbf{X})^{-1} \left(\sum_{g=1}^{m} \mathbf{X}_g'\hat{\mathbf{v}}_g\hat{\mathbf{v}}_g'\mathbf{X}_g\right) (\mathbf{X}'\mathbf{X})^{-1}$$

Cluster-robust standard errors

- CRSE do not change our estimates $\hat{\beta}$, cannot fix bias
- Valid under **clustered dependence** when main variable is constant within cluster
  - Relies on independence between clusters
  - Allows for arbitrary dependence within clusters
  - CRSEs usually $>$ conventional SEs—use when you suspect clustering
- When X_{ig} not constant within cluster, but just correlated $\rightsquigarrow$ more complicated.
  - See Abadie, Athey, Imbens, and Wooldridge (2021).
- Consistency of the CRSE are in the number of groups, not the number of individuals
  - CRSEs can be incorrect with a small ($<$ 50 maybe) number of clusters