2:Conditional Probability

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Overview

1. Conditional Probability
2. Bayes's Rule
3. Independence

Conditional probability

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- Conditional probability is the cornerstone of quantitative social science.

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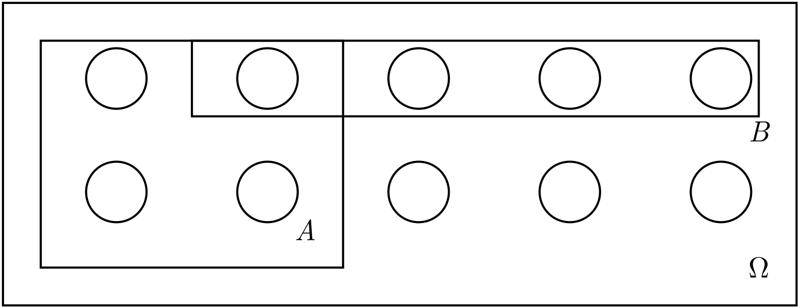
  - Also known as the **prosecutor's fallacy**.

Intuition



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  - The intersection $A \cap B = \emptyset$, so that $\mathbb{P}(A \mid B) = 0$.
  - Intuitively, it's because B occurring precludes A from occurring.

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- Not for the right-hand side, so even if B and C are disjoint,

$$\mathbb{P}(A \mid B \cup C) \neq \mathbb{P}(A \mid B) + \mathbb{P}(A \mid C).$$

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- Generalize to the intersection of N events:

$$\mathbb{P}(A_1, \ldots, A_N) = \mathbb{P}(A_1)\mathbb{P}(A_2 \mid A_1)\mathbb{P}(A_3 \mid A_1, A_2) \ldots \mathbb{P}(A_N \mid A_1, \ldots, A_{N-1}).$$

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- Thus,

$$\mathbb{P}(\text{Ace}_1 \cap \text{Ace}_2 \cap \text{Ace}_3) = \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \approx 0.00018.$$

Probability of War Resolution

- Suppose we observed country-dyads over 3 years.
- In each year the dyad can be at war (W_t) or at peace (P_t).
- What's the probability that a war starts in year 1 and ends after 2 years?

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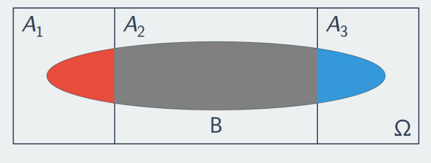
- **Actual Research Question™:** modeling the continuation probability of war, $\mathbb{P}(W_2 \mid W_1)$, and the probability of conflict resolution, $\mathbb{P}(P_3 \mid W_1, W_2)$.

Law of Total Probability



- Often we only have disaggregated probabilities.
  - $B =$ sampling a Trump supporter from either Cambridge or Somerville.
  - We know the prop. of Trump supporters in each city from precinct data.
  - How to calculate the overall probability of B?
- A **partition** is a set of mutually disjoint events whose union is Ω.

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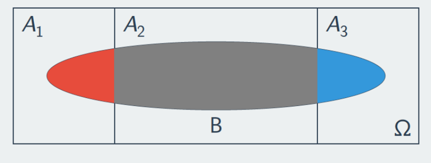

- The **law of total probability** (LTP) states if $A_1, \dots, A_k$ is a partition:

$$\mathbb{P}(B) = \sum_{j=1}^{k} \mathbb{P}(B \mid A_j)\mathbb{P}(A_j).$$

  - Overall probability = weighted sum of within-partition probabilities.
  - Weights are the probability of the particular partition.

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  - $\mathbb{P}(\text{Trump} \mid \text{Camb.}) = 0.066$
  - $\mathbb{P}(\text{Trump} \mid \text{Somer.}) = 0.103$.
- To get the overall turnout rate, $\mathbb{P}(\text{Trump})$, we can apply the LTP:

$$\mathbb{P}(\text{Trump}) = \mathbb{P}(\text{Trump} \mid \text{Camb.})\mathbb{P}(\text{Camb.}) + \mathbb{P}(\text{Trump} \mid \text{Somer.})\mathbb{P}(\text{Somer.})$$

$$= 0.066 \times 0.56 + 0.103 \times 0.44 = 0.082.$$

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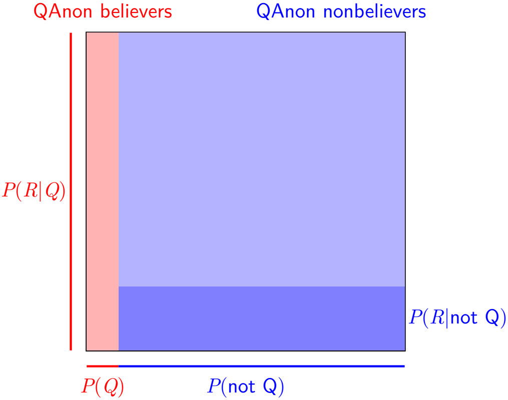
- Common response: probably believes in QAnon since believers tend to be Republicans.
- **Base rate fallacy**: ignores how uncommon QAnon believers are!

Visualizing QAnon support



Chance a random Republican believes QAnon $= \frac{P(R|Q)P(Q)}{P(R|Q)P(Q)+P(R|\text{not Q})P(\text{not Q})}$

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  - $\mathbb{P}(\text{Republican} \mid \text{QAnon})$
- Combine this with the **prior probability** of the hypothesis.

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  - $\mathbb{P}(C) = 0.007$ rough prevalence of active COVID cases.

Applying Bayes' rule to COVID tests

- Use the law of total probability to get the denominator:

$$\mathbb{P}(PT) = \mathbb{P}(PT \mid C)\mathbb{P}(C) + \mathbb{P}(PT \mid C^c)\mathbb{P}(C^c)$$

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- If false positive rate goes up to 1% $\rightsquigarrow \mathbb{P}(C \mid PT) \approx 0.36$

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  - Works other way too: if $\mathbb{P}(A) > 0$ and $A \perp\!\!\!\perp B$, then $\mathbb{P}(B \mid A) = \mathbb{P}(B)$.

Independence

- **Common misunderstanding: independent is different than disjoint!**
  - Mutually exclusive events **provide** information!

Independence example

- If we have a gathering of size n drawn randomly from the population of MA with a current COVID infection rate of 1.37%, what's the probability someone in attendance is infected?
- When seeing "prob. of at least one" $\rightsquigarrow$ work with complement:

$$\mathbb{P}(\text{At least one COVID case at gathering})$$

$$= 1 - \mathbb{P}(\text{No COVID cases at gathering})$$

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  - $n = 100 \rightsquigarrow$ prob of 0.5

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  - Ind. $\not\Rightarrow$ cond. ind.: test scores, athletics, and college admission.