1:Basic Probability

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What is Probability?

- What is a reasonably safe gathering size in the age of COVID?

Probability and Assumptions

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  - Are the data consistent with probabilities?
  - If not, assumptions are doubtful $\Rightarrow$ we've learned something.

Conjunction Fallacy

- Why mathematical probability? Our intuitions about probability are often terrible.

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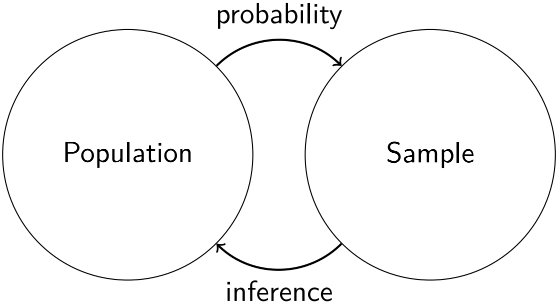
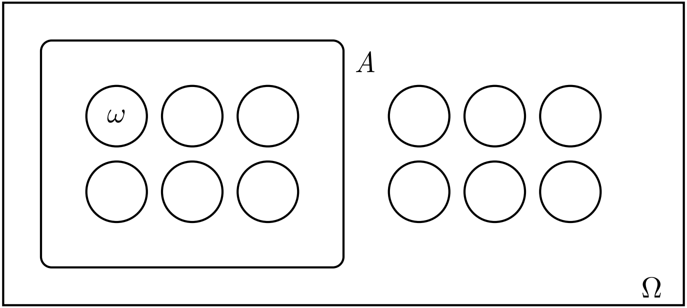
- What is more probable?
  - Linda is a bank teller?
  - Linda is a bank teller and active in the feminist movement?
- Majority chose the second, but this is impossible (conjunction fallacy).
- Learning mathematical probability avoids such mistakes.

Learning About Populations



- **Probability:** formalize the uncertainty about how our data came to be.
- **Inference:** learning about the population from a set of data.

Sample Spaces and Events



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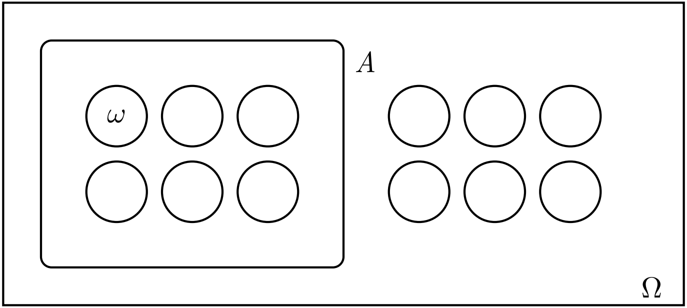

- Probability formalizes chance variation or uncertainty.
- Definitions:
  - Sample space Ω: Set of possible outcomes.
  - Event A: Subset of Ω.
- Example: Finite, countably infinite, or uncountably infinite sample spaces.

Naive Definition of Probability

- All outcomes equally likely:

$$P(A) = \frac{\text{Number of elements in } A}{\text{Number of elements in } \Omega}$$

- Justifications:
  - Symmetry: Fair coin, shuffled cards, etc.
  - Simple random samples. Ω
- Example: Probability of drawing a specific card from a deck.
  - Each card has equal probability:

$$P(4\clubsuit) = P(4\heartsuit) = \frac{1}{52}$$

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- Assumes number of outcomes in one experiment doesn't depend on the outcome of the other experiment.

Sampling objects

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- **Ex:** 2020 IA caucuses had 11 candidates. How many top-3 possibilities?
  - 11 first-place choices
  - 10 second-place choices among the remaining candidates
  - 9 third-place choices
  - Total: $11 \cdot 10 \cdot 9 = 990$ possibilities

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    ways to arrange them.
- **Binomial coefficient:** number of subsets of size k in a group of n objects:

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

The lady tasting tea

Lady tasting tea

Your friend asks you to grab a tea with milk for her before meeting up and she says that she prefers tea poured before the milk. You stop by a local tea shop and ask for a tea with milk. When you bring it to her, she complains that it was prepared milk-first.

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  - Prepare 8 cups of tea, 4 milk-first, 4 tea-first.
  - Present cups to friend in a **random** order.
  - Ask friend to pick which 4 of the 8 were milk-first.

Assuming we know the truth

- The lady picks out all 4 milk-first cups correctly!
- Statistical thought experiment: how often would she get all 4 correct **if she were guessing randomly?**
  - Only one way to choose all 4 correct cups.
- Chances of guessing all 4 correct is

$$\frac{1}{70} \approx 0.014 \text{ or } 1.4\%.$$

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Birthday Problem

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There are k people in a room. Assume each person's birthday is equally likely to be any of the 365 days of the year (no leap babies) and birthdays are independent. What is the probability that at least one pair of people have the same birthday?

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- Probability function assigns "mass" to regions of the sample space.

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     - ✓ How likely I think a particular event will be.
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  - ▶ Properties of probabilities exactly the same in either approach.
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Gambling 102

- What's the probability of selecting a 4 card from a well-shuffled deck?
  - "Well-shuffled" $\rightsquigarrow$ "randomly selected" $\rightsquigarrow$ all cards have prob. $1/52$.
- "4 card" event $= \{4\clubsuit \cup 4\spadesuit \cup 4\heartsuit \cup 4\diamondsuit\}$
- Union of mutually exclusive events $\rightsquigarrow$ use additivity:

$$\mathbb{P}(4 \text{ card}) = \mathbb{P}(4\clubsuit) + \mathbb{P}(4\spadesuit) + \mathbb{P}(4\heartsuit) + \mathbb{P}(4\diamondsuit) = \frac{4}{52}.$$

Some properties of probabilities

1. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
   - Probability of not A is 1 minus the probability of A.
   - Follows from $A \cup A^c = \Omega$ and $1 = \mathbb{P}(\Omega) = \mathbb{P}(A) + \mathbb{P}(A^c)$.
2. If $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$
   - Subsets of events have lower probability than the event.
   - Probability of 5 SCOTUS votes in the liberal direction is less than the probability of liberals winning the case ($A = \{5, 6, 7, 8, 9\}$).
3. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
   - Avoid "double-counting" the part where A and B overlap.
   - **Inclusion-exclusion**

Gambling

- A standard deck of playing cards has 52 cards:
  - 13 rank cards: (2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A)
  - In each of 4 suits: $(\clubsuit, \spadesuit, \heartsuit, \diamondsuit)$
- Hypothetical experiment: pick a card, any card.
- One possible outcome: picking a 4$\clubsuit$.
- Sample space:
  2$\clubsuit$ 3$\clubsuit$ 4$\clubsuit$ 5$\clubsuit$ . . . A$\clubsuit$
  2$\spadesuit$ 3$\spadesuit$ 4$\spadesuit$ 5$\spadesuit$ . . . A$\spadesuit$
  2$\heartsuit$ 3$\heartsuit$ 4$\heartsuit$ 5$\heartsuit$ . . . A$\heartsuit$
  2$\diamondsuit$ 3$\diamondsuit$ 4$\diamondsuit$ 5$\diamondsuit$ . . . A$\diamondsuit$
- An event: picking a Queen, $\{Q\clubsuit, Q\spadesuit, Q\heartsuit, Q\diamondsuit\}$

Social science examples

- Examples of substantively interesting **sample spaces**:
  - House elections: incumbents win or lose $\Omega = \{W, L\}$.
  - Supreme Court votes in a liberal direction: $\Omega = \{0, 1, \ldots, 9\}$.
  - Voter turnout: percent turnout in some county: $\Omega = [0, 100]$.
  - Duration of a war: any nonnegative number, $\Omega = [0, \infty)$.
- **Events** from these examples:
  - House election: incumbent wins, $A = W$.
  - Supreme Court: liberals win a Supreme Court case, $A = \{5, 6, 7, 8, 9\}$.
  - Voter turnout: over half the population votes, $A \in (50, 100]$.
  - Duration of war: war ends within five years, $A \in [0, 5]$.
- We want to know or model the probability of these events!

New events from old events

- Define events/sets $A, B \subseteq \Omega$.
- **Complement:** A^c ("not A")
  - All of the outcomes in Ω not in A.
  - Complement of picking a red card is picking a **black card**.
  - $\Omega^c = \emptyset$, where $\emptyset$ is the empty set—nothing happens.
- **Union** of two events $A \cup B$ ("A or B")
  - The event that A or B occurs.
  - Queen $\cup\clubsuit =$ any club card or any queen card.
- **Intersection** $A \cap B$ or just AB ("A and B")
  - The event that both A and B occur.
  - Queen $\cap\clubsuit = Q\clubsuit$ (queen of clubs).
- **De Morgan's laws:** $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.
  - not $Q \cup \clubsuit$ must be not Q and not $\clubsuit$.
  - not $Q \cap \clubsuit = Q\clubsuit$ is either not a Q or not a $\clubsuit$.

Relationships between events

- A **implies** B when $A \subseteq B$.
  - $A = 4\clubsuit$ and $B = \clubsuit$ card.
- **Disjoint/mutually exclusive:** $A \cap B = \emptyset$.
  - $A = \clubsuit$ card, $B = \spadesuit$ card.
- **A partition** is a set of mutually disjoint events whose union is Ω.
  - $A_1 = \clubsuit, A_2 = \spadesuit, A_3 = \diamondsuit, A_4 = \heartsuit$
  - A_1, A_2, A_3, A_4 is a partition of a 52-card deck.