3: Random Variables

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  - What is the true Biden approval rate in the US?
- Today: given a probability distribution, what data is likely?
  - If we knew the true Biden approval, what samples are likely?

Roadmap

1. Random variables
2. Famous distributions
3. Cumulative distribution functions
4. Functions of random variables
5. Independent random variables

What are Random Variables?

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- Randomness comes from the randomness of the experiment.

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- Usually abstract away from the underlying sample space fairly quickly.

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- The **support** of X is the values x such that

$$\mathbb{P}(X = x) > 0.$$

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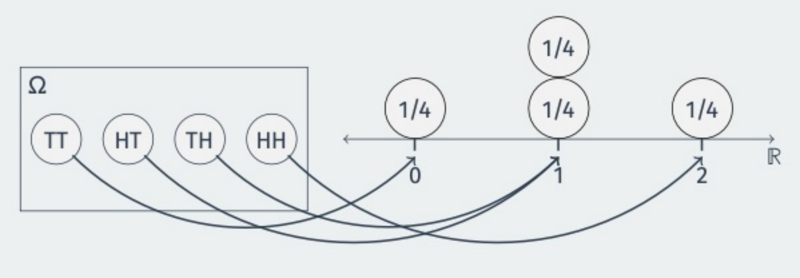
- Often there are many ways to express a distribution.

Inducing probabilities



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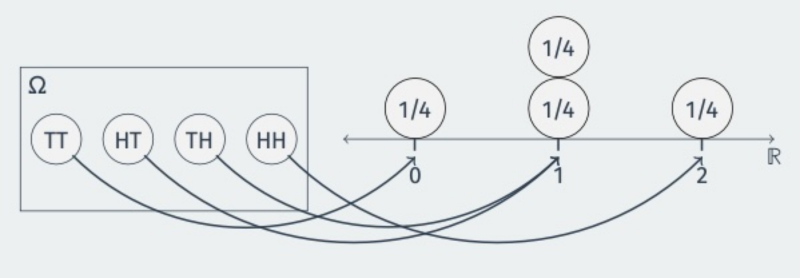

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- Probability of a set of values $S \subset \{x_1, x_2, \dots\}$:

$$\mathbb{P}(X \in S) = \sum_{x \in S} p_X(x)$$

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$$X = \begin{cases} 0 & \text{if } (C, C, C) \\ 1 & \text{if } (T, C, C) \text{ or } (C, T, C) \text{ or } (C, C, T) \\ 2 & \text{if } (T, T, C) \text{ or } (C, T, T) \text{ or } (T, C, T) \\ 3 & \text{if } (T, T, T) \end{cases}$$

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$$\mathbb{P}(C, T, C) = \mathbb{P}(C)\mathbb{P}(T)\mathbb{P}(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

Calculating the p.m.f.

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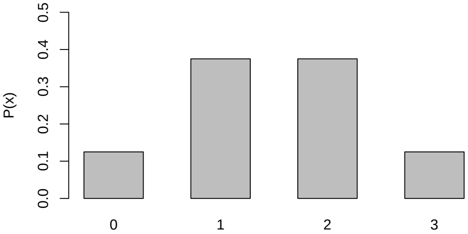
Plotting the p.m.f.

- We could plot this p.m.f. using R:



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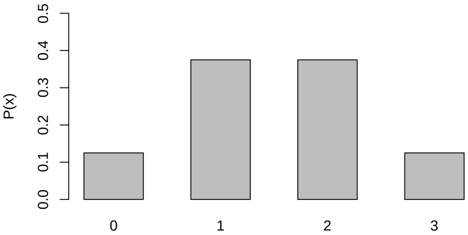

- **Question:** Does this seem like a good way to assign treatment? What is one major problem with it?

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- Story: indicator of success in some trial with either success or failure.
- Actually a **family** of distributions indexed by p.
- Any event A has an associated Bernoulli r.v.: **indicator variable** $\mathbb{I}(A) \sim \text{Bern}(p)$ with $p = P(A)$

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- Also helps draw immediate connections:
  - $\text{Bin}(1, p) \sim \text{Bern}(p)$.
  - If $X \sim \text{Bin}(n, p)$, then $n - X \sim \text{Bin}(n, 1 - p)$.

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If $X \sim \text{Bin}(n, p)$, then the p.m.f. of X is

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k},$$

for all $k = 0, 1, \ldots, n$.

- $p^k(1-p)^{n-k}$ is the probability of a **specific** sequence of 1's and 0's with k 1's.

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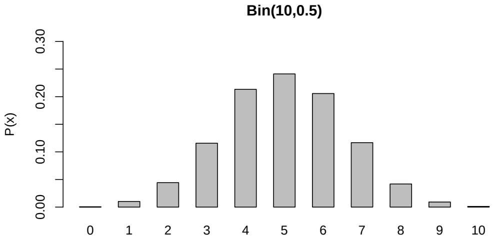
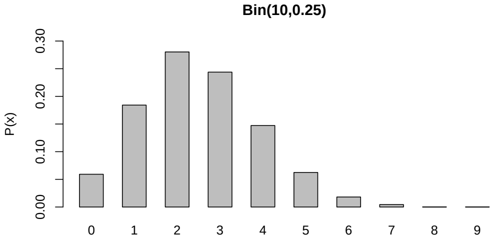
- Binomial coefficient $\binom{n}{k}$ is how many of these combinations there are.

Some Binomial Distribution





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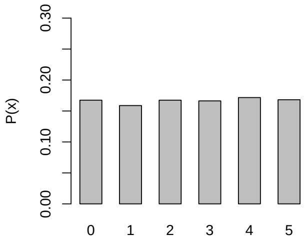
- p.m.f. for a discrete uniform r.v.:

$$p_X(x) = \begin{cases} \frac{1}{|C|}, & \text{for } x \in C \\ 0, & \text{otherwise} \end{cases}$$



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- For discrete r.v.:

$$F_X(x) = \sum_{x_j \leq x} p_X(x_j)$$

Example of discrete c.d.f

- Remember example where X is the number of treated units:

x	$\mathbb{P}(X = x)$
0	$1/8$
1	$3/8$
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$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1/8 & x \in \{0\} \\ 1/2 & x \in \{0, 1\} \\ 7/8 & x \in \{0, 1, 2\} \\ 1 & x \leq 3 \end{cases}$$

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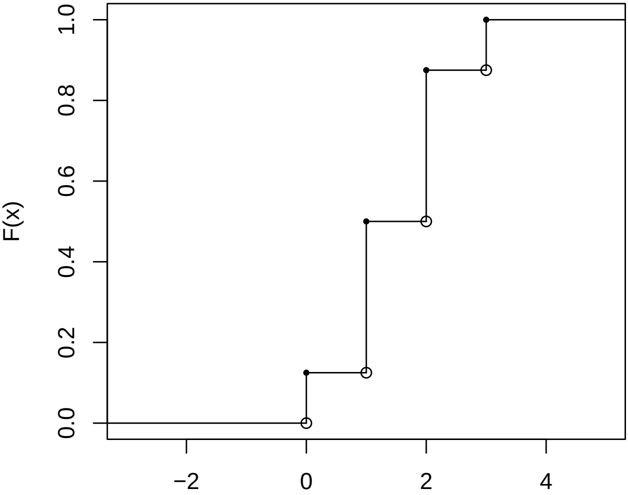
Graph of discrete c.d.f.



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- If there are redundancies, we have to add those probabilities together:

$$\mathbb{P}(Y = y_j) = \mathbb{P}(g(X) = y_j) = \sum_{x_i : g(x_i) = y_j} \mathbb{P}(X = x_i).$$

Sum vs mean vs any

- $X \sim \text{Bin}(n, p)$: number of successes.
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z	$\mathbb{P}(Z = z)$
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1	$3/8 + 3/8 + 1/8 = 7/8$

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- A few common examples:
  - If X and Y have the same distribution $\not\Rightarrow \mathbb{P}(X = Y) = 1$
  - Scaling an r.v. doesn't scale the p.m.f., so $Y = 2X$ does not have $p_Y(y) \neq 2p_X(x)$

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- For discrete r.v.s (not continuous), we can write this as:

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- **Theorem:** If $X \sim \text{Bin}(n, p)$ and $Y \sim \text{Bin}(m, p)$ with X and Y independent, then

$$X + Y \sim \text{Bin}(n + m, p).$$

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Since X and Y are independent, their joint probability is:

$$P(X = k, Y = j) = P(X = k) P(Y = j).$$

Proof: Computing the Distribution of $X + Y$

We seek to find $P(X + Y = r)$, i.e., the probability that the sum of X and Y equals r:

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Substituting the PMFs:

$$P(X + Y = r) = \sum_{k=0}^{r} \binom{n}{k} p^k (1 - p)^{n-k} \cdot \binom{m}{r - k} p^{r-k} (1 - p)^{m-(r-k)}.$$

Proof: Recognizing the Binomial Form

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Conclusion

This is exactly the PMF of a **Binomial** distribution with parameters $(n+m, p)$:

$$X + Y \sim \text{Bin}(n+m, p).$$

□

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  - $\bar{X} = \frac{1}{n}\sum_i X_i$ is our estimate of p. Properties?