4: Expectation

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Road map

- We've defined random variables and their distributions.
- Distributions give full information about the probabilities of an r.v.
- Today: begin to summarize distributions with expectation.

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- How should we summarize the distribution of causal effects?

How can we summarize distributions?

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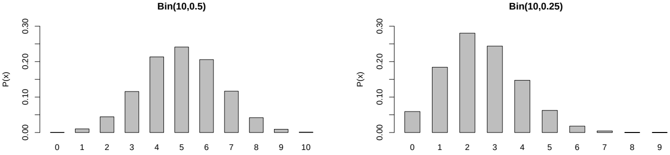

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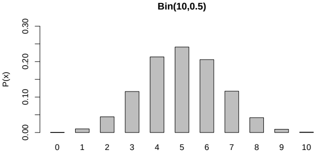

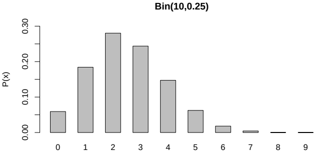

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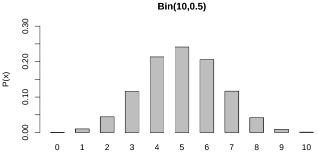
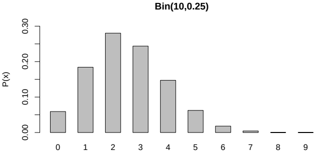
- **Question:** What is the difference between these two p.m.f.s? How might we summarize this difference?





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  - but we'll use our sample to learn about them.

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  - We'll use this intuition to create an average/mean for r.v.s.

Expectation

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The **expected value** (or **expectation** or **mean**) of a discrete r.v. X with possible values, $x_1, x_2, \ldots$ is

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  - Converse isn't true!

Example - number of treated units

- Randomized experiment with 3 units. X is the number of treated units.

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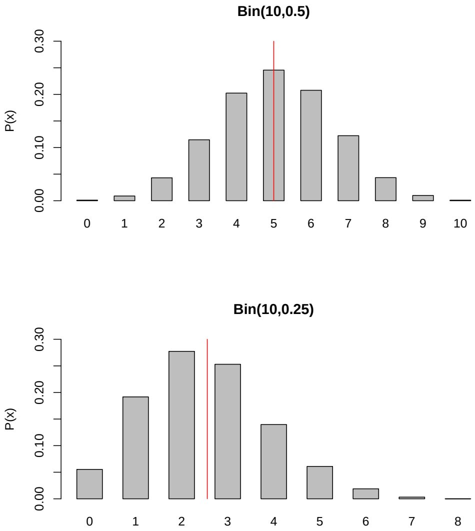
- Calculate the expectation of X:

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{j=1}^{k} x_j \mathbb{P}(X = x_j) \\
&= 0 \cdot \mathbb{P}(X = 0) + 1 \cdot \mathbb{P}(X = 1) + 2 \cdot \mathbb{P}(X = 2) + 3 \cdot \mathbb{P}(X = 3) \\
&= 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{12}{8} = 1.5
\end{aligned}$$

Expectation as balancing point



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- Use linearity:

$$\mathbb{E}[X] = \mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = np$$

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- **Intuition**: on average, the sample mean is equal to the population mean.

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  - If $X \geq Y$ with probability 1, then $\mathbb{E}(X) \geq \mathbb{E}(Y)$.

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  - Probability of reaching $X = k$ is:

$$\begin{aligned}\mathbb{P}(X = k) &= \mathbb{P}(T_1 \cap T_2 \cap \cdots \cap T_{k-1} \cap H_k) \\ &= \mathbb{P}(T_1)\mathbb{P}(T_2)\ldots\mathbb{P}(T_{k-1})\mathbb{P}(H_k) \\ &= \frac{1}{2^k}\end{aligned}$$

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  - Risk avoidance/concave utility $U = Y^{1/2} \Rightarrow \mathbb{E}[U(Y)] \approx 2.41$

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- Example: X takes 2^k and -2^k each with prob 2^{-k-1}.

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} 2^k 2^{-k-1} - \sum_{k=1}^{\infty} 2^k 2^{-k-1} = \sum_{k=1}^{\infty} \frac{1}{2} - \sum_{k=1}^{\infty} \frac{1}{2} = \infty - \infty$$

Undefined expectations

- We saw $\mathbb{E}[X]$ can be infinite, but it can also be undefined.
- Example: X takes 2^k and -2^k each with prob 2^{-k-1}.

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} 2^k 2^{-k-1} - \sum_{k=1}^{\infty} 2^k 2^{-k-1} = \sum_{k=1}^{\infty} \frac{1}{2} - \sum_{k=1}^{\infty} \frac{1}{2} = \infty - \infty$$

- Often, both of these are assumed away by assuming $\mathbb{E}[|X|] < \infty$ which implies $\mathbb{E}[X]$ exists and is finite.

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- Use the fact that $\mathbb{I}(A_1 \cup \cdots \cup A_n) \leq \mathbb{I}(A_1) + \cdots + \mathbb{I}(A_n)$ and then take expectations.

Proof of Bonferroni's Inequality

Bonferroni's Inequality: For any finite set of events $A_1, A_2, \ldots, A_n$, we have:

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) \leq P(A_1) + P(A_2) + \cdots + P(A_n)$$

Proof: Using indicator variables.

- Define the indicator random variable for each event:

$$I(A_i) = \begin{cases} 1, & \text{if } A_i \text{ occurs} \\ 0, & \text{otherwise} \end{cases}$$

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- Since $E[I(A_i)] = P(A_i)$, we get:

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- Suppose we are assigning n units to k treatments and all possibilities are equally likely. What is the expected number of treatment conditions without any units?

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$$\begin{aligned}
\mathbb{E}[I_j] &= \mathbb{P}(\text{cond } j \text{ empty}) \\
&= \mathbb{P}(\{\text{unit 1 not in cond } j\} \cap \cdots \cap \{\text{unit } n \text{ not in cond } j\}) \\
&= \mathbb{P}(\{\text{unit 1 not in cond } j\}) \dots \mathbb{P}(\{\text{unit } n \text{ not in cond } j\}) \\
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- Thus, we have :

$$\mathbb{E}\left[\sum_j I_j\right] = k\left(1 - \frac{1}{k}\right)^n.$$

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- Useful equivalent representation of the variance:

$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

LOTUS

- How do we calculate $\mathbb{E}[X^2]$ since it's nonlinear?

Definition

The **Law of the Unconscious Statistician**, or LOTUS, states that if $g(X)$ is a function of a discrete random variable, then

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Example - number of treated units

- Use LOTUS to calculate the variance for a discrete r.v.: $\mathbb{V}[X] = \sum_{j=1}^{k} (x_j - \mathbb{E}[X])^2 \mathbb{P}(X = x_j)$

x	$p_X(x)$	$x - \mathbb{E}[X]$	$(x - \mathbb{E}[X])^2$
0	1/8	-1.5	2.25
1	3/8	-0.5	0.25
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- Let's go back to the number of treated units to figure out the variance of the number of treated units:

$$\begin{aligned}
\mathbb{V}[X] &= \sum_{j=1}^{k}(x_j - \mathbb{E}[X])^2 p_X(x_j) \\
&= (-1.5)^2 \times \frac{1}{8} + (-0.5)^2 \times \frac{3}{8} + (0.5)^2 \times \frac{3}{8} + (1.5)^2 \times \frac{1}{8} \\
&= 2.25 \times \frac{1}{8} + 0.25 \times \frac{3}{8} + 0.25 \times \frac{3}{8} + 2.25 \times \frac{1}{8} = 0.75
\end{aligned}$$

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   - But this doesn't hold for dependent r.v.s.
4. $\mathbb{V}[X] \geq 0$ with equality holding only if X is a constant, $\mathbb{P}(X = b) = 1$.

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- Binomials are the sum of **independent** Bernoulli r.v.s so:

$$V[X] = V[X_1 + \cdots + X_n] = V[X_1] + \cdots + V[X_n] = np(1-p)$$

Variance of the sample mean

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- Under i.i.d. sampling we know the expectation and variance of $\bar{X}_n$ without any other assumptions about the distribution of the X_i!
  - We don't know what distribution it takes though!

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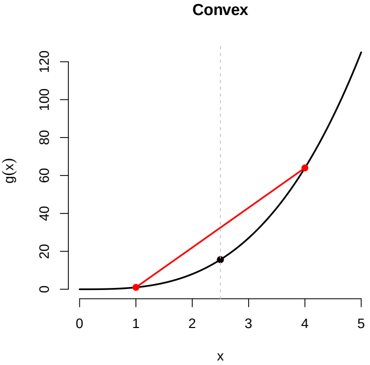
- Remember that $\mathbb{E}[a + bX] = a + b\mathbb{E}[X]$ is linear, but $\mathbb{E}[g(X)] \neq g(\mathbb{E}[X])$ for nonlinear functions.
- Can we relate those? Yes, for **convex** and **concave** functions.

Convex and Concave

Definition (Convex Function). A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *convex* if for all $x, y \in \mathbb{R}^n$ and for all $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

This means that the line segment connecting any two points on the graph of f lies above or on the graph.



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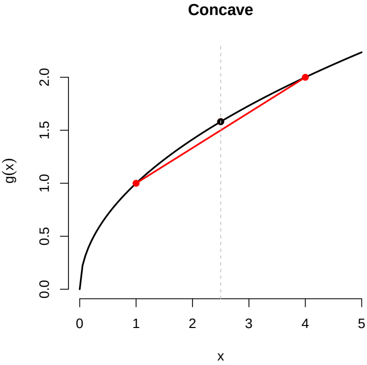

Jensen's inequality

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Let X be a r.v. Then, we have

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with equality only holding if g is linear.

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  - $\mathbb{E}[1/X] \geq 1/\mathbb{E}[X]$
  - $\mathbb{E}[\log(X)] \leq \log(\mathbb{E}[X])$

Poisson

Definition

An r.v. X has the **Poisson distribution** with parameter $\lambda > 0$, written $X \sim \text{Pois}(\lambda)$ if the p.m.f. of X is:

$$\mathbb{P}(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

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  - Number of contributions a candidate for office receives in a day.

Poisson

Definition

An r.v. X has the **Poisson distribution** with parameter $\lambda > 0$, written $X \sim \text{Pois}(\lambda)$ if the p.m.f. of X is:

$$\mathbb{P}(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

- One more discrete distribution is very popular, especially for counts.
  - Number of contributions a candidate for office receives in a day.
- Key calculus fact that makes this a valid p.m.f.: $\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}$.

Poisson properties

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- If $X \sim \text{Bin}(n, p)$ with n large and p small, then X is approximately $\text{Pois}(np)$.

Poisson Distribution as a Limiting Case of Binomial

Claim: If $X_n \sim \text{Bin}(n, p_n)$ with $n \to \infty$ and $p_n \to 0$ such that $np_n = \lambda$, then:

$$\lim_{n\to\infty} P(X_n = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots$$

which is the **Poisson** distribution $X \sim \text{Pois}(\lambda)$.

Proof:

- The binomial probability mass function (p.m.f.) is:

$$P(X_n = k) = \binom{n}{k} p_n^k (1 - p_n)^{n-k}$$

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- Using the factorial form:

$$P(X_n = k) = \frac{n!}{k!(n-k)!} p_n^k (1 - p_n)^{n-k}$$

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- Thus, X_n converges to a Poisson(λ) random variable.

Expectation of a Poisson Distribution

Claim: If $X \sim \text{Pois}(\lambda)$, then its expectation is:

$$E[X] = \lambda.$$

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- The expectation of X is given by:

$$E[X] = \sum_{k=0}^{\infty} kP(X = k).$$

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- Substituting:

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- Factor out λ:

$$E[X] = \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}.$$

Expectation of a Poisson Distribution

- Recognizing the power series for e^λ:

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = e^\lambda.$$

- Thus, we obtain:

$$E[X] = \lambda e^{-\lambda} \cdot e^\lambda = \lambda.$$

- **Conclusion:** The mean of a Poisson-distributed random variable is λ.