Section 5: Forward-Backward Propagation

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- What is the forward-backward propogation?
  - forward propagation: the algorithm that pushes our inputs through the hidden layers (weight matrices, activation functions) NN to get the final outputs
  - backward-propagation: the algorithm that finds the best-suited weight matrices that would minimize the errors in the final outputs.

Roadmap for today

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- A schematic review of what NN is doing

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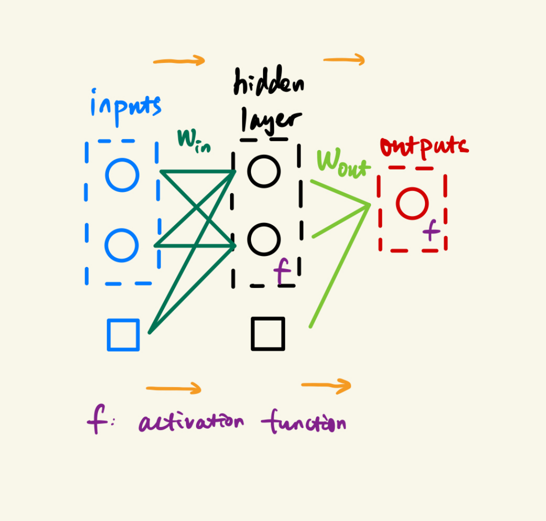
- Some coding exercise (`Rmd` file available on course website)

Feed-Forward Network

- The feed-forward neural network is a basic setup for a NN



More on Hidden Layers

Specifics

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- Hidden layers are intermediate design matrices between inputs and outputs (Deep learning is just a neural network with multiple hidden layers)
- d features, then $d + 1$ input nodes
- k possible classes, then $k - 1$ output nodes
- h nodes in the hidden layer plus an intercept, where each of these nodes is a linear combination of the inputs, passed through an activation function (though note the intercept is always active and doesn't depend on nodes in the previous layer)

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- Notice that this is the inverse of the logit function:

$$\text{logit}(x) = \log\left(\frac{x}{1-x}\right)$$

, where x is probability of an event, and $\text{logit}(x)$ is its log odds.

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Forward Propagation

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5. Put it all together: $\hat{\mathbf{y}} = \sigma(\mathbf{H}\mathbf{W}_{\text{out}}) = \sigma\left(\begin{bmatrix} 1 & \sigma(\mathbf{X}\mathbf{W}_{\text{in}}) \end{bmatrix} \mathbf{W}_{\text{out}}\right)$

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$$\mathcal{L} = \sum_i \left(y_i \log \hat{y}_i + (1 - y_i) \log (1 - \hat{y}_i) \right)$$

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- Optimize $-\mathcal{L} = f(\mathbf{W})$ via gradient descent by iterating the formula: where $\mathbf{W}_{t+1} = \mathbf{W}_t - \gamma \cdot \nabla f(\mathbf{W}_t)$ $\mathbf{W}$ and γ is 'learning rate'.

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, where

$$\frac{\partial \hat{\mathbf{y}}}{\partial \mathbf{H}} = \sigma\left(\mathbf{H}\mathbf{W}_{\text{out}}\left(1 - \sigma\left(\mathbf{H}\mathbf{W}_{\text{out}}\right)\right)\right) \mathbf{W}_{\text{out}}^\top = \hat{\mathbf{y}}(1 - \hat{\mathbf{y}})\mathbf{W}_{\text{out}}^\top$$

$$\frac{\partial \mathbf{H}}{\partial \mathbf{W}_{\text{in}}} = \mathbf{X}^\top \left[\ 1 \quad \sigma\left(\mathbf{X}\mathbf{W}_{\text{in}}\right)\left(1 - \sigma\left(\mathbf{X}\mathbf{W}_{\text{in}}\right)\right)\ \right]$$

Putting Everything Together

Training a NN using forward-backward propagation

1. Initialize weights
2. Propagate forward to get output estimates
3. Propagate error backward to update the weights toward a better solution
4. Iterate forward and back propagation until stopping criterion

- Coding exercise available on the course website (`Rmd`)